
A representation and
an interpolation theory
for harmonic Bergman functions

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1 harmonic Bergman space b^p

$$1 \leq p < \infty$$

$\Omega \subset \mathbb{R}^n$: smooth bounded domain (i.e. $\partial\Omega$ is C^∞)

$$b^p(\Omega) := \{f : \text{harmonic on } \Omega \text{ and } \|f\|_p < \infty\}$$

b^p is called the **harmonic Bergman space**.

(where $\|f\|_p := (\int_\Omega |f|^p dx)^{\frac{1}{p}}$)

Basic properties;

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- $\forall x \in \Omega, \forall f \in b^p(\Omega)$

$$f(x) = \int_\Omega R(x, y) f(y) dy$$

$R(\cdot, \cdot)$ is called **harmonic Bergman kernel**.

- Example of the harmonic Bergman kernel

The case $\Omega = \mathbb{B}$

$$\begin{aligned} R_B(x, y) &= \frac{(n-4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x \cdot y + |x|^2|y|^2)^{1+\frac{n}{2}}} \\ &= \frac{(n-4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)((1 - |x|^2)(1 - |y|^2) + |x - y|^2)^{1+\frac{n}{2}}} \end{aligned}$$

Theorem A (H. Kang and H. Koo [6] 2002)

Ω : smooth bounded domain, α, β : multi-indices.

$\exists C_{\alpha, \beta}$ s.t. $\forall x, \forall y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x, y)| \leq \frac{C_{\alpha, \beta}}{d(x, y)^{n+|\alpha|+|\beta|}}$$

$\exists C$ s.t. $\forall x \in \Omega$

$$R(x, x) \geq \frac{C}{r(x)^n}$$

where $d(x, y) := r(x) + r(y) + |x - y|$

We denote the **harmonic Bergman projection** by

$$Pf(x) := \int_{\Omega} R(x, y) f(y) dy \quad f \in L^p(\Omega)$$

If $1 < p < \infty$, then $P : L^p(\Omega) \rightarrow b^p(\Omega)$: bounded linear operator.

We obtain dualities for $1 < p < \infty$,

$$b^p(\Omega)^* \cong b^q(\Omega)$$

where q is an exponential conjugate and we consider the pairing

$$\langle f, g \rangle = \int_{\Omega} f(x) g(x) dx.$$

2 atomic decomposition for the case $p > 1$

Theorem 1 . [9]

$$1 < p < \infty$$

Ω : smooth bounded domain.

$$\exists \{\lambda_i\} \subset \Omega \text{ s.t. } \forall f \in b^p(\Omega) \exists \{a_i\} \in l^p$$

$$f(x) = \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{\frac{n}{q}}$$

where $r(x) := \text{dist}(x, \partial\Omega)$ and the convergence of series is b^p -convergence.

Remark.

$$\|R(\cdot, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\| \approx 1.$$

3 modified harmonic Bergman kernel

η : defining function for Ω s.t. $|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$

We denote the differential operator K_1 by

$$K_1 g := g - \frac{1}{2} \Delta(\eta^2 g),$$

and we denote the following;

$$R_1(x, y) := K_1(R_x)(y) \quad \text{modified harmonic Bergman kernel}$$

$$P_1 f(x) := \int_{\Omega} R_1(x, y) f(y) dy$$

.

Theorem B (B. R. Choe, H. Koo and H. Yi [2] 2004)

- $\forall f \in b^1(\Omega) \quad P_1 f = f$
- $1 \leq p < \infty \Rightarrow P_1 : L^p(\Omega) \rightarrow b^p(\Omega)$ is bounded.
- α : multi-indices $\exists C_\alpha > 0$ s.t.

$$|D_x^\alpha R_1(x, y)| \leq \frac{C_\alpha r(y)}{d(x, y)^{n+1+|\alpha|}}$$

$$|D_y^\alpha R_1(x, y)| \leq \frac{C_\alpha}{d(x, y)^{n+1}}$$

Theorem 2.

$$1 \leq p < \infty$$

Ω : smooth bounded domain.

$$\exists \{\lambda_i\} \subset \Omega \text{ s.t. } \forall f \in b^p(\Omega) \exists \{a_i\} \in l^p$$

$$f(x) = \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}$$

where the series convergence is b^p -convergence.

4 harmonic Bloch space

In this section, we assume $0 \in \Omega$.

Definition (harmonic Bloch space)

$\mathcal{B} := \{f : \text{harmonic and } \|f\|_{\mathcal{B}} < \infty\}$: harmonic Bloch space

$\tilde{\mathcal{B}} := \{f \in \mathcal{B} : f(0) = 0\}$

where $\|f\|_{\mathcal{B}} := \sup\{r(x)|\nabla f(x)|; x \in \Omega\}$ is called Bloch norm.

Properties

- $(b^1)^* \simeq \mathcal{B}$
- For $1 < \forall p < \infty$, $\mathcal{B} \subset b^p$
- $b^\infty \subset \mathcal{B}$: continuous

Theorem 3.

Ω : smooth bounded domain.

$\exists \{\lambda_i\} \subset \Omega$ s.t. $\forall f \in \tilde{\mathcal{B}} \exists \{a_i\} \in l^\infty$

$$f(x) = \sum_{i=1}^{\infty} a_i \tilde{R}_1(x, \lambda_i) r(\lambda_i)^n$$

where $\tilde{R}_1(x, y) := R_1(x, y) - R_1(0, y)$.

5 outline of the proof for atomic decompositions

We introduce the expression to describe the small interval of $\{\lambda_i\}$.

Definition (uniformly finite intersection)

A open set family $\{U_i\}$ is called uniformly finite intersection with bound

$$N \stackrel{\text{def}}{\Leftrightarrow} \exists N \text{ s.t. } \forall x \in \Omega \ \#\{i \in \mathbb{N}; x \in U_i\} \leq N$$

Lemma 1. (covering lemma)

Let $0 < \delta < \frac{1}{4}$.

We can choose N (independ of δ), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$.

(a) $\forall i \in \mathbb{N}$ E_i is measurable set

(b) $\forall i \in \mathbb{N}$ $E_i \subset B(\lambda_i, \delta r(\lambda_i))$

(c) $\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound N

Lemma 2.

$$I_\alpha f(x) := \int_{\Omega} \frac{r(y)^\alpha}{d(x, y)^{n+\alpha}} f(y) dy$$

If $\alpha = 0$, then $I_\alpha : L^p \rightarrow L^p$: bounded for $p > 1$.

If $\alpha > 0$, then $I_\alpha : L^p \rightarrow L^p$: bounded for $p \geq 1$.

proof of theorem 1

We put $0 < \delta < \frac{1}{4}$ (fixed later), $\{\lambda_i\} \subset \Omega$ and $\{E_i\}$ by lemma 1.

We consider the following operators;

$$A_{p, \{\lambda_i\}}(\{a_i\})(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{\frac{n}{q}} \text{ in } b^p$$

$$S_{p, \{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i| \text{ in } b^p$$

$$U_{p, \{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-\frac{n}{q}}\}_i$$

We change theorem 1 for

Find a condition that $A_{p,\{\lambda_i\}} : \ell^p \rightarrow b^p(\Omega)$ is bounded!

- $A_{p,\{\lambda_i\}} \circ U_{p,\{\lambda_i\}} = S_{p,\{\lambda_i\}}$
- $S_{p,\{\lambda_i\}} : b^p \rightarrow b^p$, $U_{p,\{\lambda_i\}} : b^p \rightarrow \ell^p$,
 $A_{p,\{\lambda_i\}} : \ell^p \rightarrow b^p$ are bounded operators.
- For enough small $\delta > 0$, $\|S_{p,\{\lambda_i\}} - I\| < 1$.

Hence $S_{p,\{\lambda_i\}} : b^p \rightarrow b^p$ is bijective.

The following is a part of calculation.

$$\begin{aligned}(I - S)f(x) &= \int_{\Omega} f(y)R(x, y)dy - \sum_{i=1}^{\infty} R(x, \lambda_i)f(\lambda_i)|E_i| \\ &= \sum_{i=1}^{\infty} \int_{E_i} f(y)(R(x, y) - R(x, \lambda_i))dy =: F_1(x) \\ &\quad + \sum_{i=1}^{\infty} \int_{E_i} (f(y) - f(\lambda_i))R(x, \lambda_i)dy =: F_2(x)\end{aligned}$$

$$\begin{aligned}
|F_1(x)| &\lesssim \sum_{i=1}^{\infty} \int_{E_i} |f(y)| |y - \lambda_i| |\nabla_y R(x, \bar{y})| dy \\
&\lesssim \delta \sum_{i=1}^{\infty} \int_{E_i} |f(y)| r(\lambda_i) \frac{1}{d(x, \bar{y})^{n+1}} dy \\
&\lesssim \delta \sum_{i=1}^{\infty} \int_{E_i} \frac{r(y)}{d(x, y)^{n+1}} |f(y)| dy \\
&= \delta I_1 |f|(x).
\end{aligned}$$

Hence, $\|F_1\|_p \lesssim \delta \|f\|_p$.

By similar calculation, we get $\|S_{p, \{\lambda_i\}} - I\| < C\delta$. $\square$

6 Interpolating sequence

Atomic decomposition problem is the existence of surjective map from ℓ^p to b^p .

In this section, we consider the map from b^p to ℓ^p .

We denote a operator $V_{p,\{\lambda_i\}} : b^p \rightarrow \ell^p$ as follow;

$$Vf := \{f(\lambda_i)r(\lambda_i)^{\frac{n}{p}}\}$$

Definition (quasihyperbolic metric)

$$\rho(x, y) := \inf_{\gamma_{xy} \in \Gamma_{xy}} \int_{\gamma_{xy}} \frac{1}{r(z)} ds(z)$$

where

$\Gamma_{xy} := \{\text{all smooth curve such that initial point is } x \text{ and final point is } y\}$

The following estimate is known;

$$\rho(x, y) \leq C_1 \log \frac{d(x, y)}{\min\{r(x), r(y)\}} + C_2$$

$$\frac{\min\{r(x), r(y)\}}{d(x, y)} \leq \exp\{-c'(\rho(x, y) - c'')\}$$

Theorem 4.

Let $1 < p < \infty$. There exists $\rho_0 > 0$ such that for any $\{\lambda_i\}$ with $\rho_0 < \rho(\lambda_i, \lambda_i)$ if $i \neq j$, $V_{p, \{\lambda_i\}} : b^p \rightarrow \ell^p$ is surjective.

outline of the proof

We assume that $\delta > 0$ is fixed enough small and $\{B(\lambda_i, \delta r(\lambda_i))\}_i$ is separated. We define $W_{p, \{\lambda_i\}} : \ell^p \rightarrow \ell^p$, $D_{p, \{\lambda_i\}}$ and $E_{p, \{\lambda_i\}}$ as follow;

$$W_{p, \{\lambda_i\}}(\{a_i\}) := \left\{ \sum_{j=1}^{\infty} a_j R(\lambda_i, \lambda_j) r(\lambda_j)^{\frac{n}{q}} r(\lambda_i)^{\frac{n}{p}} \right\}_i$$

$$D_{p, \{\lambda_i\}}(\{a_i\}) := \{a_i R(\lambda_i, \lambda_i) r(\lambda_i)^n\}_i$$

$$E_{p, \{\lambda_i\}}(\{a_i\}) := \left\{ \sum_{j \neq i} a_j R(\lambda_i, \lambda_j) r(\lambda_j)^{\frac{n}{q}} r(\lambda_i)^{\frac{n}{p}} \right\}_i$$

By standard argument, we should only show $\|E\| < \|D\|$.

We obtain $C \leq \|D\|$ by the behavior of the kernel.

Hence, we calculate $\|E\{a_i\}\|_{\ell^p}$. The trouble calculation's result is the following estimate.

$$\|E\{a_i\}\|_{\ell^p}^p \lesssim \sum_{i=1}^{\infty} |a_i|^p r(\lambda_i)^{\frac{1}{q}} \int_{\cup_{j \neq i} B(\lambda_j, \delta r(\lambda_j))} \frac{dy}{d(\lambda_i, y)^n r(y)^{\frac{1}{q}}}.$$

We can control the right side by using the quasihyperbolic metric.

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