A representation for harmonic Bergman function and its application

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Outline of talk

- Introduction
- Representation theorem
- Interpolation theorem
- Modified harmonic Bergman kernel
- Application

Definition and basic properties

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain.
$b^p(\Omega) := \{f : \text{ harmonic in } \Omega \text{ and } \|f\|_p < \infty\}$
b^p is called harmonic Bergman space.

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- $f \in b^2(\Omega)$ has the following representation (reproducing property);

$$f(x) = \int_\Omega R(x,y)f(y)dy \text{ for } x \in \Omega$$

$R(\cdot,\cdot)$ is called **harmonic Bergman kernel.**

Example of the harmonic Bergman kernel

When $\Omega = B$ (unit ball),

$$R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{n|B|((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+\frac{n}{2}}}$$

and

$$R_B(x,x) = \frac{(n-4)|x|^4 + 2n|x|^2 + n}{n|B|(1-|x|^2)^n}$$

The preceding result

Theorem (Kang-Koo 2002)

Let Ω be a smooth bounded domain and α and β be multi-indices. Then, there exist $C_{\alpha,\beta} > 0$ and $C > 0$ such that for any $x, y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x,y)| \leq \frac{C_{\alpha,\beta}}{d(x,y)^{n+|\alpha|+|\beta|}}$$

and

$$R(x,x) \geq \frac{C}{r(x)^n}$$

where $d(x,y) := r(x) + r(y) + |x-y|$ and $r(x)$ is the distance between x and boundary of Ω.

Harmonic Bergman projection

In the following talk, we assume that Ω is a bounded smooth domain. Then, for any $1 \leq p < \infty$, $f \in b^p(\Omega)$ has the reproducing property, that is

$$f(x) = \int_\Omega R(x,y)f(y)dy \text{ for } f \in b^p(\Omega).$$

We denote the **harmonic Bergman projection** P by

$$Pf(x) := \int_\Omega R(x,y)f(y)dy \quad f \in L^p(\Omega)$$

If $1 < p < \infty$, then $P : L^p(\Omega) \to b^p(\Omega)$ is bounded linear operator.

Representation theorem

Theorem (T. 2012)

Let $1 < p < \infty$ and Ω be a bounded smooth domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A : \ell^p \to b^p$ is a bounded onto map, where the operator A is defined by

$$A\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where $r(x)$ denotes the distance between x and $\partial\Omega$.

Outline of the proof

Lemma (covering lemma)

Let $0 < \delta < \frac{1}{4}$. We can choose N (independ of δ), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$ for Ω.

- *E_i is measurable set for any $i \in \mathbb{N}$;*
- *$E_i \subset B(\lambda_i, \delta r(\lambda_i))$ for any $i \in \mathbb{N}$;*
- *$\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound N*

Outline of the proof

We define the operators $U_{p,\{\lambda_i\}} : b^p \to \ell^p$ and $S_{p,\{\lambda_i\}} : b^p \to b^p$ as following;

$$S_{p,\{\lambda_i\}}f(x) := \sum_{i=1}^{\infty} R(x,\lambda_i)f(\lambda_i)|E_i|$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i|f(\lambda_i)r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i$$

where $\{E_i\}_i$ is the disjoint covering of Ω such that $\lambda_i \in E_i$ for any $i \in \mathbb{N}$. Because $S = A \circ U$, we may show that S is bijective map. By calculating $\|S - Id\|$, we can give the condition that S is bijective.

Definition for interpolation

In the previous section, we discussed the map from ℓ^p to b^p. In this section, we discuss the map from b^p to ℓ^p.

Definition

Let $1 < p < \infty$ and $\{\lambda_i\}_i \subset \Omega$. We define the map $V : b^p \to \ell^p$ by $V(f) = \{f(\lambda_i)r(\lambda_i)^{\frac{n}{p}}\}$

Definition (pseudo hyperbolic distance)

$$\rho(x,y) = \inf \int_{\gamma_{x,y}} \frac{1}{r(z)} ds(z)$$

where $\gamma_{x,y}$ is the C^∞-curve from an initial point x to an end point y.

Interpolation

Theorem

Let $1 < p < \infty$ and Ω be a bounded smooth domain. There exists a positive constant ρ_0 such that if $\rho(\lambda_i, \lambda_j) > \rho_0$ for any $i \neq j$, then $V : b^p \to \ell^p$ is bounded onto map, where $\rho(x, y)$ is pseudo-hyperbolic distance and $Vf := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}_i$.

Outline of the proof of interpolation

We consider

$$W\{a_i\} = V \circ A\{a_i\} = \{r(\lambda_j)^{\frac{n}{p}} \sum_i a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

We may only show W is bijective. And we define the diagonal part D and off-diagonal part E

$$D\{a_i\} := \{a_j R(\lambda_j, \lambda_j) r(\lambda_j)^n\}_j$$

and

$$E\{a_i\} := \{r(\lambda_j) \sum_{i \neq j} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

By standard argument, we may show

$$\|E\| < \frac{1}{\|D^{-1}\|}.$$

Modified harmonic Bergman kernel

We choose a defining function η for Ω such that $|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$. We denote the differential operator K_1 by

$$K_1 g := g - \frac{1}{2}\Delta(\eta^2 g),$$

and we denote the following kernel and projection;

$R_1(x, y) := K_1(R_x)(y)$: modified harmonic Bergman kernel,

where $R_x(\cdot) := R(x, \cdot)$

$$P_1 f(x) := \int_\Omega R_1(x, y) f(y) dy \quad \text{modified projection.}$$

Some property

Theorem (Choe-Koo-Yi 2004)

- $P_1 f = f$ *for any* $f \in b^1(\Omega)$.
- $P_1 : L^p(\Omega) \to b^p(\Omega)$ *is bounded for any* $1 \leq p < \infty$.
- *For any multi-index* α, *there exists* $C_\alpha > 0$ *such that*

$$|D_x^\alpha R_1(x,y)| \leq \frac{C_\alpha r(y)}{d(x,y)^{n+1+|\alpha|}}$$

$$|D_y^\alpha R_1(x,y)| \leq \frac{C_\alpha}{d(x,y)^{n+1}}$$

Modified representation

Theorem (T. (to appear in Osaka Journal))

Let $1 \leq p < \infty$ *and* Ω *be a smooth bounded domain. Then, we can choose a sequence* $\{\lambda_i\}$ *in* Ω *such that* $A_1 : \ell^p \to b^p$ *is a bounded onto map, where the operator* A_1 *is defined by*

$$A_1\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

Definition and problem for Toeplitz operator

We consider the positive Toeplitz operator on b^2.

Definition (Toeplitz operator)

We call the operator T_μ on b^2 the Toeplitz operator with symbol μ, if

$$T_\mu f(x) := \int_\Omega R(x, y) f(y) d\mu(y).$$

Problem.
What condition is the Toeplitz operator T_μ **good** (bonded, compact and Schatten σ-class S^σ etc) ?

Definition of associate functions

Definition (averaging function, Berezin transform)

For any $0 < \delta < 1$ and $1 < p < \infty$, we define

$$\hat{\mu}_\delta(x) := \frac{|\mu(E_\delta(x))|}{V(E_\delta(x))} : \text{ averaging function}$$

and

$$\tilde{\mu}_p(x) := \frac{\int_\Omega |R(x,y)|^p d\mu(y)}{\int_\Omega |R(x,y)|^p dy} : \text{ Berezin transform}$$

for any $x \in \Omega$.

The preceding result for Toeplitz operator

Theorem (Choe-Lee-Na 2004)

Let $1 \leq \sigma < \infty$ *and* $0 < \delta < 1$. *For* $\mu \geq 0$, *the following conditions are equivalent;*

1. $T_\mu \in S_\sigma$,
2. $\tilde{\mu}_2 \in L^\sigma(dV_R)$,
3. $\hat{\mu}_\delta \in L^\sigma(dV_R)$,
4. $\sum_j \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$.

for some $\{\lambda_j\}$ *satisfied with covering lemma, where* $dV_R = R(x, x)dx$.

Extension of the previous theorem

Theorem (T. (to appear in Osaka Journal))

Let $\sigma > \frac{2(n-1)}{n+2}$ and $\mu \geq 0$. Choose a constant $\delta > 0$ and a sequence $\{\lambda_j\}$ satisfying the conditions obtained by covering lemma. If $\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$, then $T_\mu \in S_\sigma$.

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