Harmonic Bergman spaces with radial measure weight on the ball

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Outline

1. Introduction and motivation
2. Harmonic Bergman space with radial measure weight
3. Toeplitz operator on harmonic Bergman space with radial measure weight
4. Bergman kernel
Let $\mathbb{B}$ be a unit ball in $\mathbb{R}^n$ and $\text{Harm}(\mathbb{B})$ be the space of all harmonic functions on $\mathbb{B}$. We define the harmonic Bergman space $b^2(\mathbb{B})$ by

$$b^2(\mathbb{B}) := \text{Harm}(\mathbb{B}) \cap L^2(\mathbb{B}, dx).$$

Moreover, for $\alpha > -1$ we consider the weighted harmonic Bergman space

$$b^2_\alpha(\mathbb{B}) := \text{Harm}(\mathbb{B}) \cap L^2(\mathbb{B}, dV_\alpha).$$
Harmonic Bergman space

When $d\nu = dx$ or $dV_\alpha$, $b^2(\mathbb{B})$ and $b^2_\alpha(\mathbb{B})$ have the following properties:

1. point evaluation map $ev_x : f \in b^2_\alpha(\mathbb{B}) \rightarrow f(x) \in \mathbb{R}$ is bounded for any $x \in \mathbb{B}$.
2. $b^2_\alpha(\mathbb{B})$: Hilbert space (complete).
3. $b^2_\alpha$ has the reproducing kernel $R_\alpha(x, y)$, i.e.,

$$f(x) = \int_{\mathbb{B}} R_\alpha(x, y)f(y)dV_\alpha(y)$$

for any $x \in \mathbb{B}$, $f \in b^2_\alpha(\mathbb{B})$.

We also consider operators on $b^2_\alpha(\mathbb{B})$. 
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\[
    f(x) = \int_{\mathbb{B}} R_\alpha(x, y)f(y)dV_\alpha(y) \text{ for any } x \in \mathbb{B}, f \in b^2_\alpha(\mathbb{B}).
\]

We also consider operators on \( b^2_\alpha(\mathbb{B}) \).
We consider the following operators:

- **Orthogonal projection** $Q_\alpha : L^2(\mathbb{B}, dV_\alpha) \rightarrow b^2_\alpha(\mathbb{B})$ which is written by

$$Q_\alpha f(x) = \int_{\mathbb{B}} R_\alpha(x, y) f(y) dV_\alpha(y) \text{ for } f \in L^2(\mathbb{B}, dV_\alpha)$$

- **Toeplitz operator** $T_{\phi, \alpha} f = Q_\alpha(\phi f)$ for $f \in b^2_\alpha(\mathbb{B})$. 
Let \( \nu \) be a positive Borel measure on \( \mathbb{B} \). We denote by

\[
b^2_\nu = b^2_\nu(\mathbb{B}) := \text{Harm}(\mathbb{B}) \cap L^2(\mathbb{B}, d\nu)\]

the harmonic Bergman space with weight \( \nu \).

Questions:

- When point evaluation maps \( f \mapsto f(x) \) on \( b^2_\alpha \) are bounded?
- When \( b^2_\nu \) is complete?
- Analysis of Toeplitz operator on \( b^2_\nu \).
Harmonic Bergman space with measure weight

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Uniqueness set


$E$ is said to be a uniqueness set for harmonic functions if a harmonic function $h$ defined on an open set $U$ satisfies $E \subset U$ and $h = 0$ on $E$, then $h = 0$ on $U$.

Remark. If $\text{supp}\nu$ is a uniqueness set for harmonic functions, then

$$\|f\|_{b^2_\nu} := \left( \int_{\mathbb{B}} |f(x)|^2 d\nu(x) \right)^{\frac{1}{2}}$$

is a norm of $b^2_\nu$. 
Let $\nu$ be a positive radial measure on $\mathbb{B}$, that is,

$$d\nu(x) = d\nu_\ast(r)d\sigma(\theta).$$

where $x = r\theta$, $r \in [0, 1)$ and $\theta \in S = \partial \mathbb{B}$.

In the case of radial measure $\nu$, $\text{supp}\nu$ is not a uniqueness set for harmonic functions if and only if $\nu = c\delta_0$ (Dirac measure at 0).

**Proposition (infinite case)**

Let $\nu$ be a positive radial Borel measure on $\mathbb{B}$. If $\nu(\mathbb{B}) = \infty$, then $b^2_{\nu} = \{0\}$.

In what follows, we assume that $\nu$ is finite.
Let $\nu$ be a positive radial measure on $\mathbb{B}$, that is,

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**Proposition (infinite case)**

Let $\nu$ be a positive radial Borel measure on $\mathbb{B}$. If $\nu(\mathbb{B}) = \infty$, then $b_\nu^2 = \{0\}$.

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Spherical harmonics

Let $\mathcal{H}_m = \mathcal{H}_m(\mathbb{R}^n)$ be the space of all harmonic homogeneous polynomials of degree $m$ and for a set $A$ in $\mathbb{R}^n$ put

$$\mathcal{H}_m(A) := \{ h|_A : h \in \mathcal{H}_m(\mathbb{R}^n) \}.$$

Let $S = \partial B$ and $\sigma$ denote by a surface measure on $S$. Then, the Hilbert space $L^2(S, \sigma) = L^2(S)$ admits the orthogonal decomposition

$$L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S).$$
Lemma (Expansion)

Let $f$ be a harmonic function on $\mathbb{B}$. Then there uniquely exist $\phi_m \in \mathcal{H}_m$ $(m \geq 0)$ such that

$$f = \sum_{m=0}^{\infty} \phi_m$$

which converges uniformly on any compact subset of $\mathbb{B}$.
Prove that $\nu(\mathbb{B}) = \infty \Rightarrow b^2_\nu = \{0\}$

By the expansion $f = \sum_{m=0}^{\infty} \phi_m \phi_m \in \mathcal{H}_m$, we have

$$\infty > \int_{\mathbb{B}} |f(x)|^2 d\nu(x) = \lim_{\epsilon \to 1} \int_0^\epsilon \int_S \left( \sum_{m=0}^{\infty} \phi_m(r\theta) \right)^2 d\sigma(\theta) d\nu_*(r)$$

$$= \lim_{\epsilon \to 1} \int_0^\epsilon \sum_{m=0}^{\infty} \int_S \phi_m(r\theta)^2 d\sigma(\theta) d\nu_*(r)$$

$$= \int_0^1 \sum_{m=0}^{\infty} r^{2m} \|\phi_m\|_{L^2(S)}^2 d\nu_*(r)$$

$$\geq \|\phi_m\|_{L^2(S)}^2 \int_0^1 r^{2m} d\nu_*(r).$$

This implies that $\phi_m = 0$ for any $m \geq 0$. 
Fundamental properties

Proposition

Let $\nu$ be a positive finite radial Borel measure on $\mathbb{B}$, i.e., $d\nu = d\nu_* d\sigma$. If $\text{supp}\nu$ is not compact on $\mathbb{B}$, that is, $\nu_*([r, 1)) > 0$ for any $r \in [0, 1)$, then the following properties holds:

- For $x \in \mathbb{B}$, $|f(x)| \leq C_x \|f\|_{b^2_\nu}$.
- $b^2_\nu$ is complete.
- $b^2_\nu$ has the reproducing kernel $R_\nu(x, y)$, i.e., for $x \in \mathbb{B}$ there exists uniquely $R_\nu(x, \cdot) \in b^2_\nu$ such that for $f \in b^2_\nu$

$$f(x) = \int_{\mathbb{B}} R_\nu(x, y) f(y) d\nu(y).$$
Proof

Let \( r_0 \in [0,1) \) and \( x \in \mathbb{B}_{r_0} \). By the Poisson formula, we have

\[
f(x) = \int_{S_r} P_r(x, \theta) f(\theta) d\sigma_r(\theta).
\]

By multiplying \( \nu_*([r_0, 1)) \) both side, we have

\[
f(x) \nu_*([r_0, 1)) = \int_{r_0}^{1} \int_{S_r} P_r(x, \theta) f(\theta) d\sigma_r(\theta) d\nu_*(r).
\]

By the estimate for Poisson kernel as follow;

\[
|P_r(x, \theta)| \leq C(|x|, r_0) \text{ for } r_0 < r < 1,
\]

we have

\[
|f(x)|^2 \nu_*([r_0, 1)) \leq C(|x|, r_0) \int_{\mathbb{B} \setminus \mathbb{B}_{r_0}} |f(x)|^2 d\nu(x) \leq C(|x|, r_0) \| f \|_{b^2}. \]
Proposition (completeness 2)

Let $\nu$ be a positive finite Borel measure on $\mathbb{B}$. Suppose $\text{supp}\nu$ be a uniqueness set for harmonic functions. If $b_2^2$ is complete, then $\nu(\mathbb{B} \setminus K) > 0$ for any compact set $K \subset \mathbb{B}$.

Therefore, we have the following result.

Theorem

Let $\nu$ be a positive finite radial Borel measure on $\mathbb{B}$ with $d\nu = d\nu_* d\sigma$. Then, $b_2^2$ is complete if and only if $\nu_*([r, 1)) > 0$ for any $r \in [0, 1)$.

We define by $M_{rad}$ the set of all positive finite radial Borel measure on $\mathbb{B}$ satisfying $\nu_*([r, 1)) > 0$ for any $r \in [0, 1)$.
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Orthogonal decomposition

Theorem (Orthogonal decomposition)

Let $\nu \in M_{\text{rad}}$. Then, we have the following orthogonal decomposition

$$b_\nu^2 = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$ 

Denote by $Z^\nu_m$ the orthogonal projection onto $\mathcal{H}_m$.

$$Z^\nu_m f(x) = \int_{\mathbb{B}} Z^\nu_m(x, y) f(y) d\nu(y)$$
**Toeplitz operator**

For a bounded measurable function \( \phi \) on \( \mathbb{B} \), we define Toeplitz operator \( T_{\phi,\nu} \) on \( b_\nu^2 \) by

\[
T_{\phi,\nu} f(x) = Q_\nu [\phi f](x) = \int_{\mathbb{B}} R_\nu(x, y) f(y) \phi(y) d\nu(y) \quad \text{for} \quad f \in b_\nu^2.
\]

**Theorem (Spectrum of Toeplitz operator)**

Let \( \nu \in M_{rad} \) and a symbol function \( \phi \) be bounded measurable. If \( \phi \) is radial, that is, \( \phi(x) = \phi_\ast(|x|) \), then the eigenvalues of the weighted Toeplitz operator \( T_{\phi,\nu} \) are the following:

\[
\lambda_m(T_{\phi,\nu}) = \frac{\int_0^1 r^{2m} \phi_\ast(r) d\nu_\ast(r)}{\int_0^1 r^{2m} d\nu_\ast(r)}
\]
Radialization of functions (Preparation for an application)

For a function $f$ on $\mathbb{B}$, we define the radicalization $\mathcal{R}f$ by

$$\mathcal{R}f(x) := \int_{U \in O(n)} R_{U^{-1}} f(x) d\lambda_{O(n)}(U)$$

for any $x \in \mathbb{B}$ where $O(n)$ is the group of orthogonal matrices and $\lambda_{O(n)}$ is the normalized Haar measure on $O(n)$ and $R_U f(x) := f(Ux)$.

$f$ is radial if and only if $\mathcal{R}f = f$. 
Radialization of operators

For an operator $T$ on $b_2^2$, we define the radicalization $\mathcal{R}T$ by

$$\mathcal{R}T = \int_{U \in O(n)} R^*_U TR_U d\lambda_{O(n)}(U)$$

where the above integral means that for $f, g \in b_2^2$,

$$\langle (\mathcal{R}T)f, g \rangle = \int_{U \in O(n)} \langle R^*_U TR_U f, g \rangle d\lambda_{O(n)}(U).$$

An operator $T$ on $b_2^2$ is called radial if $\mathcal{R}T = T$.

Let $\nu \in M_{rad}$. If a function $\phi$ is bounded measurable on $\mathbb{B}$, then

$$\mathcal{R}T_{\phi, \nu} = T_{\mathcal{R}\phi, \nu}.$$ 

In particular, $\phi$ is radial, then $T_{\phi, \nu}$ is radial.
Radialization of operators

For an operator $T$ on $b^2_\nu$, we define the radicalization $\mathcal{R}T$ by

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In particular, $\phi$ is radial, then $T_{\phi, \nu}$ is radial.
Compactness of Toeplitz operator

By the basic properties for functional analysis, we can check the following lemma.

**Lemma**

Let $\nu$ be positive finite measure on $\mathbb{B}$ and $b_\nu^2$ be complete. Assume $\phi$ is positive and belong to $C(\overline{\mathbb{B}})$. If $\phi = 0$ on $\partial \mathbb{B}$, then $T_{\phi,\nu}$ is compact.

Conversely, we have the following result.

**Theorem (compactness)**

Let $\nu \in M_{\text{rad}}, \phi$ be positive and belong to $C(\overline{\mathbb{B}})$. If the Toeplitz operator $T_{\phi,\nu}$ is compact, then $\phi = 0$ on $\partial \mathbb{B}$.
Compactness of Toeplitz operator

By the basic properties for functional analysis, we can check the following lemma.

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Conversely, we have the following result.

**Theorem (compactness)**

Let \( \nu \in M_{\text{rad}} \), \( \phi \) be positive and belong to \( C(\mathbb{B}) \). If the Toeplitz operator \( T_{\phi,\nu} \) is compact, then \( \phi = 0 \) on \( \partial \mathbb{B} \).
Lemma

Let \( \nu \in M_{\text{rad}} \). Then, we have

\[
R_\nu(x, y) = \sum_{m=0}^{\infty} Z_m^\nu(x, y) = \sum_{m=0}^{\infty} \frac{Z_m(x, y)}{(2m + n) \int_0^1 r^{2m} d\nu_*(r)}
\]

for any \( x, y \in \mathbb{B} \).
Proposition

Let $\mu$ and $\nu$ belong to $M_{rad}$. If $\mu(A) \leq \nu(A)$ for any Borel set $A \subset \mathbb{B}$, then

$$R_\mu(x, x) \geq R_\nu(x, x)$$

Moreover, the boundary behavior of $R_\nu(x, x)$ depends on that of measure $\nu$.

Theorem

Let $\mu$ and $\nu$ belong to $M_{rad}$. If

$$0 < \liminf_{r \to 1} \frac{\mu_*([r, 1))}{\nu_*([r, 1))} \leq \limsup_{r \to 1} \frac{\mu_*([r, 1))}{\nu_*([r, 1))} < \infty,$$

then

$$R_\mu(x, x) \approx R_\nu(x, x).$$
Future works

- Relation between spectrum $\lambda_m(T_{\phi,\nu})$ and averaging function $\hat{\phi}_\delta$.
- Relation between spectrum $\lambda_m(T_{\phi,\nu})$ and Berezin transform $\tilde{\phi}$.
- Estimate for $R_{\nu}(x, y)$. 


