Harmonic Bergman spaces with radial measure weight on the ball

Kiyoki Tanaka

Osaka City University Advanced Mathematical Institute (OCAMI)

Sep, 4, 2014 / Potential theory symposium

joint work with Masaharu Nishio(Osaka City Univ.)

Outline

1. Introduction and motivation
2. Harmonic Bergman space with radial measure weight
3. Toeplitz operator on harmonic Bergman space with radial measure weight
4. Bergman kernel

Harmonic Bergman space

Let $\mathbb{B}$ be a unit ball in $\mathbb{R}^n$ and $Harm(\mathbb{B})$ be the space of all harmonic functions on $\mathbb{B}$. We define the harmonic Bergman space $b^2(\mathbb{B})$ by

$$b^2(\mathbb{B}) := Harm(\mathbb{B}) \cap L^2(\mathbb{B}, dx).$$

Moreover, for $\alpha > -1$ we consider the weighted harmonic Bergman space

$$b^2_\alpha(\mathbb{B}) := Harm(\mathbb{B}) \cap L^2(\mathbb{B}, dV_\alpha).$$

Harmonic Bergman space

When $d\nu = dx$ or dV_α, $b^2(\mathbb{B})$ and $b^2_\alpha(\mathbb{B})$ have the following properties;

1. point evaluation map $ev_x : f \in b^2_\alpha(\mathbb{B}) \to f(x) \in \mathbb{R}$ is bounded for any $x \in \mathbb{B}$.
2. $b^2_\alpha(\mathbb{B})$: Hilbert space (complete).
3. b^2_α has the reproducing kernel $R_\alpha(x, y)$, i.e.,

$$f(x) = \int_{\mathbb{B}} R_\alpha(x, y) f(x) dV_\alpha(y) \text{ for any } x \in \mathbb{B}, f \in b^2_\alpha(\mathbb{B}).$$

We also consider operators on $b^2_\alpha(\mathbb{B})$.

Harmonic Bergman space

When $d\nu = dx$ or dV_α, $b^2(\mathbb{B})$ and $b^2_\alpha(\mathbb{B})$ have the following properties;

1. point evaluation map $ev_x : f \in b^2_\alpha(\mathbb{B}) \to f(x) \in \mathbb{R}$ is bounded for any $x \in \mathbb{B}$.
2. $b^2_\alpha(\mathbb{B})$: Hilbert space (complete).
3. b^2_α has the reproducing kernel $R_\alpha(x, y)$, i.e.,

$$f(x) = \int_{\mathbb{B}} R_\alpha(x, y) f(x) dV_\alpha(y) \text{ for any } x \in \mathbb{B}, f \in b^2_\alpha(\mathbb{B}).$$

We also consider operators on $b^2_\alpha(\mathbb{B})$.

Operators on harmonic Bergman space

We consider the following operators:

- orthogonal projection $Q_\alpha : L^2(\mathbb{B}, dV_\alpha) \to b^2_\alpha(\mathbb{B})$ which is written by

$$Q_\alpha f(x) = \int_{\mathbb{B}} R_\alpha(x, y) f(y) dV_\alpha(y) \text{ for } f \in L^2(\mathbb{B}, dV_\alpha)$$

- Toeplitz operator $T_{\phi,\alpha} f = Q_\alpha(\phi f)$ for $f \in b^2_\alpha(\mathbb{B})$.

Harmonic Bergman space with measure weight

Let ν be a positive Borel measure on $\mathbb{B}$.We denote by

$$b_\nu^2 = b_\nu^2(\mathbb{B}) := Harm(\mathbb{B}) \cap L^2(\mathbb{B}, d\nu)$$

the harmonic Bergman space with weight ν.

Questions:

- When point evaluation maps $f \mapsto f(x)$ on b_α^2 are bounded?
- When b_ν^2 is complete?
- Analysis of Toeplitz operator on b_ν^2.

Harmonic Bergman space with measure weight

Let ν be a positive Borel measure on $\mathbb{B}$.We denote by

$$b^2_\nu = b^2_\nu(\mathbb{B}) := Harm(\mathbb{B}) \cap L^2(\mathbb{B}, d\nu)$$

the harmonic Bergman space with weight ν.

Questions:

- When point evaluation maps $f \mapsto f(x)$ on b^2_α are bounded?
- When b^2_ν is complete?
- Analysis of Toeplitz operator on b^2_ν.

Uniqueness set

Definition (c.f. Nakazi-Yamada Can. J. Math.(1996))

E is said to be a uniqueness set for harmonic functions if a harmonic function h defined on an open set U satisfies $E \subset U$ and $h = 0$ on E, then $h = 0$ on U.

Remark. If *supp*ν is a uniqueness set for harmonic functions, then $\|f\|_{b^2_\nu} := \left(\int_{\mathbb{B}} |f(x)|^2 d\nu(x)\right)^{\frac{1}{2}}$ is a norm of b^2_ν.

Let ν be a positive radial measure on $\mathbb{B}$, that is,

$$d\nu(x) = d\nu_*(r)d\sigma(\theta).$$

where $x = r\theta$, $r \in [0, 1)$ and $\theta \in S = \partial\mathbb{B}$.
In the case of radial measure ν, $supp\nu$ is not a uniqueness set for harmonic functions if and only if $\nu = c\delta_0$(Dirac measure at 0).

Proposition (infinite case)

Let ν be a positive radial Borel measure on $\mathbb{B}$.
If $\nu(\mathbb{B}) = \infty$, then $b^2_\nu = \{0\}$.

In what follows, we assume that ν is finite.

Let ν be a positive radial measure on $\mathbb{B}$, that is,

$$d\nu(x) = d\nu_*(r)d\sigma(\theta).$$

where $x = r\theta$, $r \in [0, 1)$ and $\theta \in S = \partial\mathbb{B}$.
In the case of radial measure ν, *supp*ν is not a uniqueness set for harmonic functions if and only if $\nu = c\delta_0$(Dirac measure at 0).

Proposition (infinite case)

Let ν be a positive radial Borel measure on $\mathbb{B}$.
If $\nu(\mathbb{B}) = \infty$, then $b^2_\nu = \{0\}$.

In what follows, we assume that ν is finite.

Let ν be a positive radial measure on $\mathbb{B}$, that is,

$$d\nu(x) = d\nu_*(r)d\sigma(\theta).$$

where $x = r\theta$, $r \in [0, 1)$ and $\theta \in S = \partial\mathbb{B}$.
In the case of radial measure ν, *supp*ν is not a uniqueness set for harmonic functions if and only if $\nu = c\delta_0$(Dirac measure at 0).

Proposition (infinite case)

Let ν be a positive radial Borel measure on $\mathbb{B}$.
If $\nu(\mathbb{B}) = \infty$, then $b^2_\nu = \{0\}$.

In what follows, we assume that ν is finite.

Spherical harmonics

Let $\mathcal{H}_m = \mathcal{H}_m(\mathbb{R}^n)$ be the space of all harmonic homogeneous polynomials of degree m and for a set A in $\mathbb{R}^n$ put

$$\mathcal{H}_m(A) := \{h|_A : h \in \mathcal{H}_m(\mathbb{R}^n)\}.$$

Let $S = \partial\mathbb{B}$ and σ denote by a surface measure on S. Then, the Hilbert space $L^2(S, \sigma) = L^2(S)$ admits the orthogonal decomposition

$$L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S).$$

Expansion of harmonic functions

Lemma (Expansion)

Let f be a harmonic function on $\mathbb{B}$. Then there uniquely exist $\phi_m \in \mathcal{H}_m$ $(m \geq 0)$ such that

$$f = \sum_{m=0}^{\infty} \phi_m$$

which converges uniformly on any compact subset of $\mathbb{B}$.

Prove that $\nu(\mathbb{B}) = \infty \Rightarrow b^2_\nu = \{0\}$

By the expansion $f = \sum_{m=0}^{\infty} \phi_m \ \phi_m \in \mathcal{H}_m$, we have

$$\begin{aligned}
\infty > \int_{\mathbb{B}} |f(x)|^2 d\nu(x) &= \lim_{\epsilon \to 1} \int_0^{\epsilon} \int_S \left(\sum_{m=0}^{\infty} \phi_m(r\theta) \right)^2 d\sigma(\theta) d\nu_*(r) \\
&= \lim_{\epsilon \to 1} \int_0^{\epsilon} \sum_{m=0}^{\infty} \int_S \phi_m(r\theta)^2 d\sigma(\theta) d\nu_*(r) \\
&= \int_0^1 \sum_{m=0}^{\infty} r^{2m} \|\phi_m\|^2_{L^2(S)} d\nu_*(r) \\
&\geq \|\phi_m\|^2_{L^2(S)} \int_0^1 r^{2m} d\nu_*(r).
\end{aligned}$$

This implies that $\phi_m = 0$ for any $m \geq 0$.

Fundamental properties

Proposition

Let ν be a positive finite radial Borel measure on $\mathbb{B}$, i.e., $d\nu = d\nu_* d\sigma$. If suppν is not compact on $\mathbb{B}$, that is, $\nu_*([r,1)) > 0$ for any $r \in [0,1)$, then the following properties holds:

- *For $x \in \mathbb{B}$, $|f(x)| \leq C_x \|f\|_{b^2_\nu}$.*
- *b^2_ν is complete.*
- *b^2_ν has the reproducing kernel $R_\nu(x,y)$, i.e., for $x \in \mathbb{B}$ there exists uniquely $R_\nu(x,\cdot) \in b^2_\nu$ such that for $f \in b^2_\nu$*

$$f(x) = \int_{\mathbb{B}} R_\nu(x,y) f(y) d\nu(y).$$

Proof

Let $r_0 \in [0, 1)$ and $x \in \mathbb{B}_{r_0}$. By the Poisson formula, we have

$$f(x) = \int_{S_r} P_r(x, \theta) f(\theta) d\sigma_r(\theta).$$

By multiplying $\nu_*([r_0, 1))$ both side, we have

$$f(x)\nu_*([r_0, 1)) = \int_{r_0}^{1} \int_{S_r} P_r(x, \theta) f(\theta) d\sigma_r(\theta) d\nu_*(r).$$

By the estimate for Poisson kernel as follow;

$$|P_r(x, \theta)| \leq C(|x|, r_0) \text{ for } r_0 < r < 1,$$

we have

$$|f(x)|^2 \nu_*([r_0, 1)) \leq C(|x|, r_0) \int_{\mathbb{B}\setminus\mathbb{B}_{r_0}} |f(x)|^2 d\nu(x) \leq C(|x|, r_0) \|f\|_{b^2_\nu}.$$

Fundamental properties

Proposition (completeness 2)

Let ν be a positive finite Borel measure on $\mathbb{B}$. Suppose suppν be a uniqueness set for harmonic functions. If b^2_ν is complete, then $\nu(\mathbb{B} \setminus K) > 0$ for any compact set $K \subset \mathbb{B}$.

Therefore, we have the following result.

Theorem

Let ν be a positive finite radial Borel measure on $\mathbb{B}$ with $d\nu = d\nu_* d\sigma$. Then, b^2_ν is complete if and only if $\nu_*([r, 1)) > 0$ for any $r \in [0, 1)$.

We define by M_{rad} the set of all positive finite radial Borel measure on $\mathbb{B}$ satisfying $\nu_*([r, 1)) > 0$ for any $r \in [0, 1)$.

Fundamental properties

Proposition (completeness 2)

Let ν be a positive finite Borel measure on $\mathbb{B}$. Suppose suppν be a uniqueness set for harmonic functions. If b^2_ν is complete, then $\nu(\mathbb{B} \setminus K) > 0$ for any compact set $K \subset \mathbb{B}$.

Therefore, we have the following result.

Theorem

Let ν be a positive finite radial Borel measure on $\mathbb{B}$ with $d\nu = d\nu_* d\sigma$. Then, b^2_ν is complete if and only if $\nu_*([r, 1)) > 0$ for any $r \in [0, 1)$.

We define by M_{rad} the set of all positive finite radial Borel measure on $\mathbb{B}$ satisfying $\nu_*([r, 1)) > 0$ for any $r \in [0, 1)$.

Orthogonal decomposition

Theorem (Orthogonal decomposition)

Let $\nu \in M_{rad}$. Then, we have the following orthogonal decomposition

$$b_\nu^2 = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

Denote by Z_m^ν the orthogonal projection onto $\mathcal{H}_m$.

$$Z_m^\nu f(x) = \int_{\mathbb{B}} Z_m^\nu(x, y) f(y) d\nu(y)$$

Toeplitz operator

For a bounded measurable function ϕ on $\mathbb{B}$, we define Toeplitz operator $T_{\phi,\nu}$ on b^2_ν by

$$\begin{aligned} T_{\phi,\nu}f(x) &= Q_\nu[\phi f](x) \\ &= \int_{\mathbb{B}} R_\nu(x,y)f(y)\phi(y)d\nu(y) \text{ for } f \in b^2_\nu. \end{aligned}$$

Theorem (Spectrum of Toeplitz operator)

Let $\nu \in M_{rad}$ and a symbol function ϕ be bounded measurable. If ϕ is radial, that is, $\phi(x) = \phi_*(|x|)$, then the eigenvalues of the weighted Toeplitz operator $T_{\phi,\nu}$ are the following;

$$\lambda_m(T_{\phi,\nu}) = \frac{\int_0^1 r^{2m}\phi_*(r)d\nu_*(r)}{\int_0^1 r^{2m}d\nu_*(r)}$$

Radialization of functions(Preparationfor an application)

For a function f on $\mathbb{B}$, we define the radicalization $\mathcal{R}f$ by

$$\mathcal{R}f(x) := \int_{U \in O(n)} R_{U^{-1}} f(x) d\lambda_{O(n)}(U)$$

for any $x \in \mathbb{B}$ where $O(n)$ is the group of orthogonal matrices and $\lambda_{O(n)}$ is the normalized Haar measure on $O(n)$ and $R_U f(x) := f(Ux)$.
f is radial if and only if $\mathcal{R}f = f$.

Radialization of operators

For an operator T on b^2_ν, we define the radicalization $\mathcal{R}T$ by

$$\mathcal{R}T = \int_{U \in O(n)} R_U^* T R_U d\lambda_{O(n)}(U)$$

where the above integral means that for $f, g \in b^2_\nu$,

$$\langle (\mathcal{R}T) f, g \rangle = \int_{U \in O(n)} \langle R_U^* T R_U f, g \rangle_\nu d\lambda_{O(n)}(U).$$

An operator T on b^2_ν is called radial if $\mathcal{R}T = T$.
Let $\nu \in M_{rad}$. If a function ϕ is bounded measurable on $\mathbb{B}$, then

$$\mathcal{R}T_{\phi,\nu} = T_{\mathcal{R}\phi,\nu}.$$

In particular, ϕ is radial, then $T_{\phi,\nu}$ is radial.

Radialization of operators

For an operator T on b^2_ν, we define the radicalization $\mathcal{R}T$ by

$$\mathcal{R}T = \int_{U \in O(n)} R_U^* T R_U d\lambda_{O(n)}(U)$$

where the above integral means that for $f, g \in b^2_\nu$,

$$\langle (\mathcal{R}T)f, g \rangle = \int_{U \in O(n)} \langle R_U^* T R_U f, g \rangle_\nu d\lambda_{O(n)}(U).$$

An operator T on b^2_ν is called radial if $\mathcal{R}T = T$.
Let $\nu \in M_{rad}$. If a function ϕ is bounded measurable on $\mathbb{B}$, then

$$\mathcal{R}T_{\phi,\nu} = T_{\mathcal{R}\phi,\nu}.$$

In particular, ϕ is radial, then $T_{\phi,\nu}$ is radial.

Compactness of Toeplitz operator

By the basic properties for functional analysis, we can check the following lemma.

Lemma

Let ν be positive finite measure on $\mathbb{B}$ and b^2_ν be complete. Assume ϕ is positive and belong to $C(\overline{\mathbb{B}})$.
If $\phi = 0$ on $\partial\mathbb{B}$, then $T_{\phi,\nu}$ is compact.

Conversely, we have the following result.

Theorem (compactness)

Let $\nu \in M_{rad}$, ϕ be positive and belong to $C(\overline{\mathbb{B}})$. If the Toeplitz operator $T_{\phi,\nu}$ is compact, then $\phi = 0$ on $\partial\mathbb{B}$.

Compactness of Toeplitz operator

By the basic properties for functional analysis, we can check the following lemma.

Lemma

Let ν be positive finite measure on $\mathbb{B}$ and b^2_ν be complete. Assume ϕ is positive and belong to $C(\overline{\mathbb{B}})$.
If $\phi = 0$ on $\partial\mathbb{B}$, then $T_{\phi,\nu}$ is compact.

Conversely, we have the following result.

Theorem (compactness)

Let $\nu \in M_{rad}$, ϕ be positive and belong to $C(\overline{\mathbb{B}})$. If the Toeplitz operator $T_{\phi,\nu}$ is compact, then $\phi = 0$ on $\partial\mathbb{B}$.

Expansion of harmonic Bergman kernel

Lemma

Let $\nu \in M_{rad}$. Then, we have

$$R_\nu(x,y) = \sum_{m=0}^{\infty} Z_m^\nu(x,y) = \sum_{m=0}^{\infty} \frac{Z_m(x,y)}{(2m+n)\int_0^1 r^{2m} d\nu_*(r)}$$

for any $x, y \in \mathbb{B}$.

Comparison of Bergman kernel

Proposition

Let μ and ν belong to M_{rad}. If $\mu(A) \leq \nu(A)$ for any Borel set $A \subset \mathbb{B}$, then

$$R_\mu(x,x) \geq R_\nu(x,x)$$

Moreover, the boundary behavior of $R_\nu(x,x)$ depends on that of measure ν.

Theorem

Let μ and ν belong to M_{rad}. If

$$0 < \liminf_{r \to 1} \frac{\mu_*([r,1))}{\nu_*([r,1))} \leq \limsup_{r \to 1} \frac{\mu_*([r,1))}{\nu_*([r,1))} < \infty, \text{ then}$$

$$R_\mu(x,x) \approx R_\nu(x,x).$$

Future works

- Relation between spectrum $\lambda_m(T_{\phi,\nu})$ and averaging function $\hat{\phi}_\delta$.
- Relation between spectrum $\lambda_m(T_{\phi,\nu})$ and Berezin transform $\tilde{\phi}$.
- Estimate for $R_\nu(x, y)$.

Reference

- S. Axler, P. Bourdon and W. Ramey, Harmonic function theory, Springer-Verlag, New York, 1992.
- S. Grudsky and N. Vasilevski *Bergman-Toeplitz operators radial component influence*, Integr. equ. oper. theory **40** (2001), 16–33.
- Y. J. Lee, *Compact radial operators on the harmonic Bergman space*, J. Math. Kyoto Univ., Vol. 44 (2004), 769–777.
- T. Nakazi and M. Yamada, *Riesz's functions in weighted Hardy and Bergman spaces*, Can. J. Math., Vol. 48 (1996), 930–945.
- T.Nakazi and R.Yoneda, *Compact Toeplitz operators with continuous symbols on weighted Bergman spaces*, Glasgow Math. J., Vol 42 (2000), 31–35.
- K. Stroethoff, *Compact Toeplitz operators on weighted harmonic Bergman spaces*, J. Austral. Math. Soc., Vol 64 (1998), 136–148.