

Representation and interpolating sequence for harmonic Bergman functions

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Outline

1 Setting

2 Results

3 Outline of the proof

Harmonic Bergman space

$1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$: bounded smooth domain.

$b^p(\Omega) := \{f : \text{harmonic in } \Omega \text{ and } \|f\|_p < \infty\}$

b^p : harmonic Bergman space.

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- $f \in b^2(\Omega)$ has a reproducing property:

$$f(x) = \int_{\Omega} R(x, y) f(y) dy \text{ for } x \in \Omega$$

$R(\cdot, \cdot)$: **harmonic Bergman kernel**.

Representation theorem

Theorem (T. 2012)

Let $1 < p < \infty$ and Ω be a bounded smooth domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A_{p,\{\lambda_i\}} : \ell^p \rightarrow b^p$ is a bounded onto map, where the operator $A_{p,\{\lambda_i\}}$ is defined by

$$A_{p,\{\lambda_i\}}\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where $r(x)$ denotes the distance between x and $\partial\Omega$.

Fix a defining function η for Ω s.t. $|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$.
 B.R. Choe, H. Koo and H. Yi (2004) introduced the following kernel

$$R_1(x, y) := R(x, y) - \frac{1}{2}\Delta_y(\eta^2(y)R_x(y)).$$

and shown that R_1 has the reproducing property.

Theorem (T. 2013)

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$$A_1\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

Interpolating sequence

Let denote $V : b^p(\Omega) \rightarrow l^p$ by

$$V_{p,\{\lambda_i\}} f := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}.$$

It is known that $A_{p,\{\lambda_i\}}^* = V_{q,\{\lambda_i\}}$ for $1 < p < \infty$, $q: \frac{1}{p} + \frac{1}{q} = 1$.

Theorem (T. (to appear in RIMS kôkyûroku bessatu))

Let $1 < p < \infty$. We can choose a positive constant ρ_0 satisfying the following condition;

if $\{\lambda_i\}_i \subset \Omega$ satisfy quasi-hyperbolic distance $\rho(\lambda_i, \lambda_j) > \rho_0$ for $i \neq j$, then $V : b^p(\Omega) \rightarrow l^p$ is bounded and onto.

$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x,y}} \int_{\gamma} \frac{1}{r(z)} ds(z)$$

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The preceding works

Bergman type spaces are studied on several setting.

$$L_a^2(\mathbb{D}, dA) = \{f : \text{holo. on } \mathbb{D}, \|f\|_{L^2} < \infty\}$$

- R.R. Coifman - R. Rochberg (1980) : holomorphic on unit ball in \mathbb{C}^n
- B.R. Choe - H. Yi (1998) : harmonic on upper half space in \mathbb{R}^n
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The preceding result

Theorem (Kang-Koo 2002)

Let Ω be a smooth bounded domain and α and β be multi-indices. Then, there exist $C_{\alpha,\beta} > 0$ and $C > 0$ such that for any $x, y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x, y)| \leq \frac{C_{\alpha,\beta}}{d(x, y)^{n+|\alpha|+|\beta|}}$$

and

$$R(x, x) \geq \frac{C}{r(x)^n}$$

where $d(x, y) := r(x) + r(y) + |x - y|$ and $r(x)$ is the distance between x and boundary of Ω .

Outline of the proof of representation theorem

Lemma (covering lemma)

Let $0 < \delta < \frac{1}{4}$. We can choose N (independ of δ), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$ for Ω .

- 1 E_i is measurable set for any $i \in \mathbb{N}$;
- 2 $E_i \subset B(\lambda_i, \delta r(\lambda_i))$ for any $i \in \mathbb{N}$;
- 3 $\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound N

Remark.

If there exists a positive constant $c > 0$ such that $\{B(\lambda_i, cr(\lambda_i))\}$ is uniformly finite intersection, then $A_{p, \{\lambda_i\}}$ and $V_{p, \{\lambda_i\}}$ are bounded for $1 < p < \infty$.

We define the operators $U_{p,\{\lambda_i\}} : b^p \rightarrow \ell^p$ and $S_{p,\{\lambda_i\}} : b^p \rightarrow b^p$ as following;

$$S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i|$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i$$

where $\{E_i\}_i$ is the disjoint covering of Ω such that $\lambda_i \in E_i$ for any $i \in \mathbb{N}$. Because $S = A \circ U$, we may show that S is bijective map. By calculating $\|S - Id\|$, we can give the condition that S is bijective.

Outline of the proof of interpolating sequence theorem

We assume that there exist $\delta > 0$ such that $\{B(\lambda_i, \delta r(\lambda_i))\}$ is disjoint.

We put

$$W_{p, \{\lambda_i\}} \{a_i\} := V \circ A \{a_i\} = \{r(\lambda_j)^{\frac{n}{p}} \sum_{i=1}^{\infty} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

We choose the $\{\lambda_i\}_i$ such that the operator W is bijective.

We put diagonal part of W

$$D \{a_i\} := \{a_i R(\lambda_j, \lambda_j) r(\lambda_j)^n\}$$

and off diagonal part of W

$$E \{a_i\} := \{r(\lambda_j)^{\frac{n}{p}} \sum_{i \neq j} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

Because D : bijective and $W = D + E$, $\|E\| < \frac{1}{\|D^{-1}\|} \Rightarrow W$: bijective.

Moreover, since $R(x, x)r(x)^n \approx 1$, $\|D\| \approx 1$. We have only to give the estimate for $\|E\|$.

$$\|E\{a_i\}\|_p \lesssim \delta^{\frac{-n(p-1)}{p}} \left(\sum_{i=1}^{\infty} |a_i|^p r(\lambda_i)^{\frac{1}{q}} \sum_{j \neq i} r(\lambda_j)^{n-\frac{1}{q}} |R(\lambda_j, \lambda_i)| \right)^{\frac{1}{p}}$$

We calculate

$$r(\lambda_i)^{\frac{1}{q}} \sum_{j \neq i} r(\lambda_j)^{n-\frac{1}{q}} |R(\lambda_j, \lambda_i)| \lesssim \int_{\Omega \setminus E_\delta(\lambda_i)} \frac{r(\lambda_i)^{\frac{1}{q}} r(z)^{-\frac{1}{q}}}{d(z, \lambda_i)^n} dz$$

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Lemma

There exist constants $C_1 > 0$ and $C_2 > 0$ such that for any $x, y \in \Omega$

$$\frac{1}{d(x, y)} \leq \frac{\exp(-\frac{\rho(x, y) - C_1}{C_2})}{\min\{r(x), r(y)\}}.$$

By using quasi-hyperbolic distance condition $\rho(\lambda_i, \lambda_j) > \rho_0$, we have

$$\int_{\Omega \setminus E_\delta(\lambda_i)} \frac{r(\lambda_i)^{\frac{1}{q}} r(z)^{-\frac{1}{q}}}{d(z, \lambda_i)^n} dz \lesssim \exp(-\epsilon(\frac{\rho_0 - C_1}{C_2})).$$

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