Operators on harmonic Bergman spaces

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Outline

1. Introduction and motivation
2. harmonic Bergman space on the unit ball
3. harmonic Bergman space on smooth bounded domain
4. Modified harmonic Bergman kernel
5. Application for Toeplitz operator

Bergman space on the unit disc

We denote the weighted Bergman space on the unit disc $\mathbb{D}$ by

$$L^2_a(\mathbb{D}, dA_\alpha) := \{f : \mathbb{D} \to \mathbb{C} : f : \text{analytic in } \mathbb{D} \text{ and } \|f\|_{2,\alpha} < \infty\}$$

where $\alpha > -1$,

$$dA_\alpha(z) := C_\alpha(1 - |r|^2)^\alpha d\theta dr$$

C_α is the normalized constant.

$$\text{norm} : \|f\|_{2,\alpha} := \left(\int_{\mathbb{D}} |f|^2 dA_\alpha\right)^{\frac{1}{2}}$$

In 1922, S. Bergman stated this space (the case $\alpha = 0$, no-weighted case).

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We have the following properties:

- $L^2_a(\mathbb{D}, dA_\alpha) \subset L^2(\mathbb{D}, dA_\alpha)$: closed subspace
- $L^2_a(\mathbb{D}, dA_\alpha)$: reproducing kernel Hilbert space, i.e., $\forall z \in \mathbb{D}\ \exists \overline{K_\alpha(z,\cdot)} \in L^2_a(\mathbb{D}, dA_\alpha)$ s.t. $\forall f \in L^2_a(\mathbb{D}, dA_\alpha)$

$$f(z) = \int_{\mathbb{D}} K_\alpha(z,w) f(w) dA_\alpha(w) \text{ (reproducing property)}$$

- $K_\alpha(z,w) = \overline{K_\alpha(w,z)}$: anti-symmetric
- $$K_\alpha(z,w) = \frac{1}{(1 - z\bar{w})^{2+\alpha}}$$

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Bergman projection, Toeplitz operator on Bergman space

We consider the orthogonal projection Q_α from $L^2(\mathbb{D}, dA_\alpha)$ to $L^2_a(\mathbb{D}, dA_\alpha)$.

It is well-known that for $f \in L^2(\mathbb{D}, dA_\alpha)$, Q_α has the form

$$Q_\alpha f(x) = \int_{\mathbb{D}} K_\alpha(x, y) f(y) dA_\alpha(y).$$

For a function φ, we denote Toeplitz operator $\mathbb{T}^{(\alpha)}_\varphi$ on $L^2_a(\mathbb{D}, dA_\alpha)$ by

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Motivation 1, Bergman metric

We put

$$H(z) := \frac{\partial^2}{2\partial z \partial \bar{z}} \log K(z, z) = \frac{1}{(1 - |z|^2)^2}.$$

Then, $\sqrt{H(z)}ds(z)$, where ds is Euclidean length element, is called Bergman metric (or Poincaré metric).

Problem

We can consider the Bergman kernel on a general domain. Then, we don't have explicit form of Bergman kernel.

How domain has good estimate?

Answer(?)

We can only answer that a "certain" domain (for example, a domain has smooth boundary, upper-half plane) is OK.

But, we don't have completely answer. (on the setting of today's talk, the kernel has estimate)

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Motivation 2, previous results on "quantization"

We consider the map $\varphi \mapsto \mathbb{T}_{\varphi}^{(\alpha)}$. This map has the following properties:

1. $\varphi \mapsto \mathbb{T}_{\varphi}^{(\alpha)}$ is linear
2. $\lim_{\alpha\to\infty} \|\mathbb{T}_{\varphi}^{(\alpha)}\| = \|\varphi\|_{\infty}$ for $\varphi, \psi \in C(\bar{\mathbb{D}})$
3. $\lim_{\alpha\to\infty} \|\alpha[\mathbb{T}_{\varphi}^{(\alpha)}, \mathbb{T}_{\psi}^{(\alpha)}] - i\mathbb{T}_{\{\varphi,\psi\}}^{(\alpha)}\| = 0$ for $\varphi, \psi \in C^{\infty}(\bar{\mathbb{D}})$ where $[\mathbb{T}_{\varphi}^{(\alpha)}, \mathbb{T}_{\psi}^{(\alpha)}] := \mathbb{T}_{\varphi}^{(\alpha)}\mathbb{T}_{\psi}^{(\alpha)} - \mathbb{T}_{\psi}^{(\alpha)}\mathbb{T}_{\varphi}^{(\alpha)}$ and $\{\varphi, \psi\} := i(1-|z|^2)^2 \left(\partial\varphi(z)\bar{\partial}\psi(z) - \bar{\partial}\varphi(z)\partial\psi(z)\right)$

In the observation as quantum physic, this relation seems to be quantization, that is,

$$\begin{array}{ccc} \{\text{Classical mechanics}\} & \longrightarrow & \{\text{quantum mechanics}\} \\ \varphi & \longrightarrow & \mathbb{T}_{\varphi}^{(\alpha)} \end{array}$$

Problem Characterize Toeplitz operators. For example, $\mathbb{T}_{\varphi}$ is bounded $\Leftrightarrow$ φ has something conditions. We discuss this problem later under the another setting (which is similar setting).

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Characterization for radial Toeplitz operators

We comment the example of result on Toeplitz operator.
Let φ be the bounded radial function (, that is, $\varphi(z) = \varphi(|z|)$) on $\mathbb{D}$. It is known the eigenvalue of $\mathbb{T}_\varphi$: $m \in \mathbb{N}_0$

$$\lambda_m(\mathbb{T}_\varphi) = \frac{\left(\int_0^1 \varphi(r) r^{2m+1}(1-r^2)^\alpha dr\right)^{\frac{1}{2}}}{\left(\int_0^1 r^{2m+1}(1-r^2)^\alpha dr\right)^{\frac{1}{2}}}.$$

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Generalization

In the following section, we consider the another setting's space. Until now, we put

$$L_a^2(\mathbb{D}, dA) := \{f : \mathbb{D} \to \mathbb{C} : f : \text{analytic in } \mathbb{D} \text{ and } \|f\|_2 < \infty\},$$

- exponent $2 \to p$.
- measure A (norm $\|\cdot\|_{2,\alpha}$) $\to$ another weighted measure.
- analytic $\to$ solutions of differential equation.
- domain $\mathbb{D} \to$ "general" domain.

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harmonic Bergman space

$1 \le p < \infty$, $\Omega \subset \mathbb{R}^n$: domain.
$b^p(\Omega) := \{f : \Omega \to \mathbb{R}$ harmonic and $\|f\|_p < \infty\}$: harmonic Bergman space

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- In particular, $b^2(\Omega) \subset L^2(\Omega)$: reproducing kernel Hilbert space
- $f \in b^2(\Omega)$ has a reproducing property: for $x \in \Omega$, there exists unique $R(x, \cdot) \in b^2(\Omega)$

$$f(x) = \int_\Omega R(x, y) f(y) dy.$$

$R(\cdot, \cdot)$: **harmonic Bergman kernel.**

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properties for harmonic Bergman kernel

We introduce the properties for harmonic Bergman kernel.

- $R(x, y)$ has real value.
- symmetric $R(x, y) = R(y, x)$.
- $\|R(x, \cdot)\|_{b^2}^2 = R(x, x)$.
- If $\{e_m(\cdot)\}_{m \in \mathbb{N}}$ is orthogonal basis of $b^2(\Omega)$, then

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For a function φ, we denote Toeplitz operator T_φ on $b^2(\Omega)$ by

$$T_\varphi f(x) := P(f\varphi)(x) = \int_\Omega R(x,y)f(y)\varphi(y)dy$$

For a measure μ, we denote Toeplitz operator T_μ on $b^2(\Omega)$ by

$$T_\mu f(x) := \int_\Omega R(x,y)f(y)d\mu(y).$$

When $\Omega = \mathbb{B}$ (unit ball),

$$R(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{n|\mathbb{B}|((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+\frac{n}{2}}}$$

and

$$R_{\mathbb{B}}(x,x) = \frac{(n-4)|x|^4 + 2n|x|^2 + n}{n|\mathbb{B}|(1-|x|^2)^n}$$

When $\Omega = \mathbb{B}$: unit ball, $b^p(\mathbb{B}) \subset b^q(\mathbb{B})$ $(1 \leq q < p \leq \infty)$.
$b^p(\mathbb{B})^* \simeq b^q(\mathbb{B})$ ($1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$)
$f \in b^p(\mathbb{B})$ $(1 \leq p \leq \infty)$ has the reproducing property

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Harmonic Bergman projection P is extended from $L^p(\mathbb{B}) \to b^p(\mathbb{B})$ $(1 < p < \infty)$ and $P : L^p(\mathbb{B}) \to b^p(\mathbb{B})$ is bounded.

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Weighted harmonic Bergman space

For $\alpha > -1$, we consider the weighted harmonic Bergman space $b^p_\alpha(\mathbb{B})$ denoted by

$$b^p_\alpha(\mathbb{B}) := \{f : \text{ harmonic on } \mathbb{B} \text{ and } \|f\|_{p,\alpha} < \infty\}$$

where

$$\|f\|_{p,\alpha} := \left(\int_{\mathbb{B}} |f(x)|^p dV_\alpha(x) \right)^{\frac{1}{p}}$$

and $dV_\alpha(x) = (1 - |x|^2)^\alpha dx$.

Weighted kernel, projection

By same method, $f \in b^2_\alpha(\mathbb{B})$ has a reproducing property: for $x \in \mathbb{B}$, there exists unique $R_\alpha(x, \cdot) \in b^2_\alpha(\mathbb{B})$

$$f(x) = \int_{\mathbb{B}} R_\alpha(x, y) f(y) dV_\alpha(y).$$

Orthogonal projection P_α form $L^2(\mathbb{B}, dV_\alpha)$ to $b^2_\alpha(\mathbb{B})$ has the form

$$P_\alpha f(x) = \int_{\mathbb{B}} R_\alpha(x, y) f(y) dV_\alpha(y).$$

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Properties for weighted harmonic Bergman kernel

Let $\alpha > 0$. Then,

- $b^p(\mathbb{B}) \subset b^p_\alpha(\mathbb{B})$.
- For $x, y \in \mathbb{B}$,

$$R_\alpha(x,x) \approx \frac{1}{(1-|x|)^{n+\alpha}}$$

$$R_\alpha(x,y) \lesssim \frac{1}{|x-y|^{n+\alpha}}$$

- $P_\alpha : L^p(\mathbb{B}) \to b^p(\mathbb{B})$ is bounded for $1 \leq p$.

$$P_\alpha f(x) = \int_{\mathbb{B}} f(y) R_\alpha(x,y)(1-|y|^2)^\alpha dy.$$

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Known results of Toeplitz operators on $b^2(\mathbb{B})$

Definition (averaging function, Berezin transform)

For any $0 < \delta < 1$,

$$\hat{\varphi}_\delta(x) := \frac{\int_{E_\delta(x)} \varphi(y)dy}{V(E_\delta(x))} : \text{averaging function}$$

$$\tilde{\varphi}(x) := \frac{\int_{\mathbb{B}} |R(x,y)|^2 \varphi(y)dy}{R(x,x)} : \textbf{Berezin transform}$$

for any $x \in \mathbb{B}$, where $E_\delta(x) := \{y \in \mathbb{B} : |x - y| < \delta(1 - |x|)\}$.

We can describe the boundedness of Toeplitz operator T_φ by using the above associate functions.

Known results of Toeplitz operators on $b^2(\mathbb{B})$

Theorem

Let φ be a positive function on $\mathbb{B}$. Then, the following conditions are equivalent:

- *T_φ is bounded;*
- *averaging function $\hat{\varphi}$ is bounded function;*
- *Berezin transform $\tilde{\varphi}$ is bounded function.*

Theorem

Let φ be a positive function on $\mathbb{B}$. Then, the following conditions are equivalent:

- *T_φ is compact;*
- *averaging function $\hat{\varphi}(x) \to 0$ as $|x| \to 1$;*
- *Berezin transform $\tilde{\varphi}(x) \to 0$ as $|x| \to 1$.*

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The preceding result

We consider Ω **is smooth bounded domain** in $\mathbb{R}^n$. We have the following theorem.

Theorem (Kang-Koo 2002)

Let Ω be a smooth bounded domain and α and β be multi-indices. Then, there exist $C_{\alpha,\beta} > 0$ and $C > 0$ such that for any $x, y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x,y)| \leq \frac{C_{\alpha,\beta}}{d(x,y)^{n+|\alpha|+|\beta|}}$$

and

$$R(x,x) \geq \frac{C}{r(x)^n}$$

where $d(x,y) := r(x) + r(y) + |x-y|$ and $r(x)$ is the distance between x and boundary of Ω.

Harmonic Bergman projection

$1 \leq p < \infty$, $f \in b^p(\Omega)$ has the reproducing property, that is

$$f(x) = \int_\Omega R(x,y) f(y) dy$$

$$Pf(x) := \int_\Omega R(x,y) f(y) dy \quad f \in L^p(\Omega)$$

harmonic Bergman projection

$1 < p < \infty \Rightarrow P : L^p(\Omega) \to b^p(\Omega)$: bounded linear operator.

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Preparation for results

We introduce the decomposition of domain Ω.

Lemma (covering lemma)

Let $0 < \delta < \frac{1}{4}$. *We can choose N (independ of* δ*),* $\{\lambda_i\} \subset \Omega$ *and disjoint covering* $\{E_i\}$ *for* Ω.

1. E_i *is measurable set for any* $i \in \mathbb{N}$;
2. $E_i \subset B(\lambda_i, \delta r(\lambda_i))$ *for any* $i \in \mathbb{N}$;
3. $\{B(\lambda_i, 3\delta r(\lambda_i))\}$ *is uniformly finite intersection with bound N*

where $r(x)$ *denotes the distance between* x *and* $\partial\Omega$.

This contraction is similar to Whitney decomposition.

Representation theorem

Theorem (T. 2012)

Let $1 < p < \infty$, Ω be a bounded smooth domain. There exists $0 < \delta_0$ such that if $\{\lambda_i\}$ satisfies covering lemma for $\delta < \delta_0$, then $A_{p,\{\lambda_i\}} : \ell^p \to b^p$ is a bounded onto map, where the operator $A_{p,\{\lambda_i\}}$ is defined by

$$A_{p,\{\lambda_i\}}\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where $r(x)$ denotes the distance between x and $\partial\Omega$.

Remark.
$\|R(x, \cdot)\|_{b^p} \lesssim r(\lambda_i)^{(1-\frac{1}{p})n}$. $R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}$ plays role of the "basis".

Interpolating of harmonic Bergman functions

Let denote $V : b^p(\Omega) \to l^p$ by

$$V_{p,\{\lambda_i\}} f := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}.$$

It is known that $A^*_{p,\{\lambda_i\}} = V_{q,\{\lambda_i\}}$ for $1 < p < \infty$, q: $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem (T. 2013)

Let $1 < p < \infty$. We can choose a positive constant ρ_0 satisfying the following condition;
if $\{\lambda_i\}_i \subset \Omega$ satisfy quasi-hyperbolic distance $\rho(\lambda_i, \lambda_j) > \rho_0$ for $i \neq j$, then $V : b^p(\Omega) \to l^p$ is bounded and onto.

$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x,y}} \int_\gamma \frac{1}{r(z)} ds(z)$$

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Outline of the proof of representation theorem

We define the operators $U_{p,\{\lambda_i\}} : b^p \to \ell^p$ and $S_{p,\{\lambda_i\}} : b^p \to b^p$ as following;

$$S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i|$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i$$

where $\{E_i\}_i$ is the disjoint covering of Ω such that $\lambda_i \in E_i$ for any $i \in \mathbb{N}$. Because $S = A \circ U$, by calculating $\|S - Id\|$, we can give the condition that S is bijective.

Modified harmonic Bergman kernel

Fix a defining function η for Ω s.t.
$|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$.
We denote the differential operator K_1 by

$$K_1 g := g - \frac{1}{2}\Delta(\eta^2 g)$$

$$R_1(x,y) := K_1(R_x)(y) \quad \text{: modified harmonic Bergman kernel,}$$

where $R_x(\cdot) := R(x,\cdot)$

$$P_1 f(x) := \int_\Omega R_1(x,y) f(y)\,dy \quad \text{modified projection.}$$

Some property

Theorem (Choe-Koo-Yi 2004)

- $P_1 f = f$ *for any* $f \in b^1(\Omega)$.
- $P_1 : L^p(\Omega) \to b^p(\Omega)$ *is bounded for any* $1 \leq p < \infty$.
- *For any multi-index* α, *there exists* $C_\alpha > 0$ *such that*

$$|D_x^\alpha R_1(x,y)| \leq \frac{C_\alpha r(y)}{d(x,y)^{n+1+|\alpha|}}$$

$$|D_y^\alpha R_1(x,y)| \leq \frac{C_\alpha}{d(x,y)^{n+1}}$$

Comparison between kernels

properties	Bergman kernel	modified kernel
symmetric	symmetric	non-symmetric
reproducing property	exist for $p \geq 1$	exist for $p \geq 1$
lower bdd	exist	not exist
upper bdd	$\frac{1}{d(x,y)^n}$	$\frac{r(y)}{d(x,y)^{n+1}}$
projection	bdd for $p > 1$	bdd for $p \geq 1$

Modified representation

Theorem (T. 2013)

Let $1 \leq p < \infty$ and Ω be a smooth bounded domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A_1 : \ell^p \to b^p$ is a bounded onto map, where the operator A_1 is defined by

$$A_1\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

Harmonic Bloch space

$$\mathcal{B} := \{f : \Omega \to \mathbb{R} : \text{ harmonic, } \|f\|_{\mathcal{B}} < \infty\}$$

$$\|f\|_{\mathcal{B}} := \sup\{r(x)|\nabla f(x)| : x \in \Omega\}$$

$$(b^1)^* \cong \mathcal{B}$$

For fix $x_0 \in \Omega$,

$$\mathcal{B}_0 := \{f \in \mathcal{B} : f(x_0) = 0\}$$

Representation for harmonic Bloch function

Theorem (T. 2013)

Ω be a smooth bounded domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A_\infty : \ell^\infty \to \mathcal{B}_0$ is a bounded onto map, where the operator A_∞ is defined by

$$f(x) = \sum_{j=1}^{\infty} a_j \tilde{R}_1(x, \lambda_j) r(\lambda_j)^n,$$

where $\tilde{R}_1(x, y) = R_1(x, y) - R_1(0, y)$.

Definition and problem for Toeplitz operator

Definition (Toeplitz operator)

T_μ on b^2 the Toeplitz operator with symbol μ

$$T_\mu f(x) := \int_\Omega R(x,y) f(y) d\mu(y).$$

Problem.
What condition is the Toeplitz operator T_μ **good** (bounded, compact and of Schatten σ-class S^σ etc) ?

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Averaging function, Berezin transform

We define the averaging function and Berezin transform on Ω which are similar to the unit ball case

Definition (averaging function, Berezin transform)

For any $0 < \delta < 1$ and $1 < p < \infty$,

$$\hat{\mu}_\delta(x) := \frac{|\mu(E_\delta(x))|}{V(E_\delta(x))} \text{ : averaging function}$$

$$\tilde{\mu}_p(x) := \frac{\int_\Omega |R(x,y)|^p d\mu(y)}{\int_\Omega |R(x,y)|^p dy} \text{ : Berezin transform}$$

for any $x \in \Omega$.

The preceding result for Toeplitz operator

Theorem (Choe-Lee-Na 2004)

Let $1 \le \sigma < \infty$ and $0 < \delta < 1$. For $\mu \ge 0$, the following conditions are equivalent;

- $T_\mu \in S_\sigma$,
- $\tilde{\mu}_2 \in L^\sigma(dV_R)$,
- $\hat{\mu}_\delta \in L^\sigma(dV_R)$,
- $\sum_j \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$.

for some $\{\lambda_j\}$ satisfied with covering lemma, where $dV_R = R(x,x)dx$.

c.f.

For T : compact operator on Hilbert space $\mathcal{H}$, $0 < \sigma < \infty$

$$T \text{ belongs to } \sigma\text{-Schatten class } S_\sigma \Leftrightarrow \sum_{m=1}^{\infty} s_m(T)^\sigma < \infty$$

where $\{s_m(T)\}_m$ is singular value sequence of T.

Extension of the previous theorem

Theorem (T. 2013)

Let $\sigma > \frac{2(n-1)}{n+2}$ *and* $\mu \geq 0$. *Choose a constant* $\delta > 0$ *and a sequence* $\{\lambda_j\}$ *satisfying the conditions obtained by covering lemma. If* $\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$, *then* $T_\mu \in S_\sigma$.

Outline of the proof

By the standard operator theory,
$X \in S^\sigma$ and Y is bdd operator $\Rightarrow XY, YX \in S^\sigma$.

- First, we check the condition $A^* T_\mu A \in S^\sigma(\ell^2)$ (by using assumption).
- $T_\mu = (US^{-1})^* A^* T_\mu A (US^{-1})$ belongs to S^σ. $\square$

$$\begin{tikzcd} b^2(\Omega) \arrow[r, "T_\mu"] \arrow[d, "(US^{-1})^*"'] & b^2(\Omega) \\ \ell^2 \arrow[r, "A^* T_\mu A"] & \ell^2 \arrow[u, "US^{-1}"'] \end{tikzcd}$$

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$$\begin{array}{ccc} b^2(\Omega) & \xrightarrow{T_\mu} & b^2(\Omega) \\ {\scriptstyle (US^{-1})^*}\big\downarrow & & \big\uparrow{\scriptstyle US^{-1}} \\ \ell^2 & \xrightarrow{A^* T_\mu A} & \ell^2 \end{array}$$

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