# A representation for harmonic Bergman functions and its applications

Kiyoki Tanaka

Department of Mathematics
Osaka City University

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# Outline

# Harmonic Bergman Space $b^p$

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ is smooth bounded domain.
$b^p(\Omega) := \{f : \text{ harmonic in } \Omega \text{ and } \|f\|_p < \infty\}$
$b^p$ is called harmonic Bergman space.

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- For any $x \in \Omega$, $f \in b^p(\Omega)$ has the following representation;

$$f(x) = \int_\Omega R(x, y) f(y) dy$$

$R(\cdot, \cdot)$ is called **harmonic Bergman kernel**.

# Example of the harmonic Bergman kernel

The case $\Omega = \mathbb{B}$ (unit ball)

$$R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x\cdot y + |x|^2|y|^2)^{1+\frac{n}{2}}}$$

$$= \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+\frac{n}{2}}}$$

# The recently result

## Theorem ( H. Kang and H. Koo 2002 )

*Let $\Omega$ be a smooth bounded domain and $\alpha$ and $\beta$ be multi-indices. Then, there exist $C_{\alpha,\beta>0}$ and $C > 0$ such that for any $x, y \in \Omega$*

$$|D_x^\alpha D_y^\beta R(x,y)| \leq \frac{C_{\alpha,\beta}}{d(x,y)^{n+|\alpha|+|\beta|}}$$

$$R(x,x) \geq \frac{C}{r(x)^n}$$

*where $d(x,y) := r(x) + r(y) + |x-y|$ and $r(x)$ is the distance between $x$ and boundary $\Omega$.*

# Harmonic Bergman projection

We denote the **harmonic Bergman projection** by

$$Pf(x) := \int_\Omega R(x, y) f(y) dy \quad f \in L^p(\Omega)$$

If $1 < p < \infty$, then $P : L^p(\Omega) \to b^p(\Omega)$ is bounded linear operator.

# Representation theorem

**Theorem (K. Tanaka (to appear in Hiroshima Journal))**

*Let $1 < p < \infty$ and $\Omega$ be a smooth bounded domain. Then, we can choose $\{\lambda_i\} \subset \Omega$ such that for any $f \in b^p(\Omega)$ there exists $\{a_i\} \in \ell^p$ such that*

$$f(x) = \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}$$

*where the convergence of series is $b^p$-convergence.*

# Modified harmonic Bergman kernel

We choose a defining function $\eta$ for $\Omega$ such that $|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$. We denote the differential operator $K_1$ by

$$K_1 g := g - \frac{1}{2}\Delta(\eta^2 g),$$

and we denote the following kernel and projection;

$R_1(x,y) := K_1(R_x)(y)$ : modified harmonic Bergman kernel,

$$P_1 f(x) := \int_\Omega R_1(x,y) f(y) dy \quad \text{modified projection.}$$

# Some property

## Theorem (B. R. Choe, H. Koo and H. Yi 2004)

- *$P_1 f = f$ for any $f \in b^1(\Omega)$.*
- *$P_1 : L^p(\Omega) \to b^p(\Omega)$ is bounded for any $1 \leq p < \infty$.*
- *For any multi-index $\alpha$, there exists $C_\alpha > 0$ such that*

$$|D_x^\alpha R_1(x, y)| \leq \frac{C_\alpha r(y)}{d(x, y)^{n+1+|\alpha|}}$$

$$|D_y^\alpha R_1(x, y)| \leq \frac{C_\alpha}{d(x, y)^{n+1}}$$

# Result 1

**Theorem ( K. Tanaka (to appear in Osaka Journal))**

*Let $1 \leq p < \infty$ and $\Omega$ be a smooth bounded domain. Then, we can choose $\{\lambda_i\} \subset \Omega$ such that for any $f \in b^p(\Omega)$ there exist $\{a_i\} \in \ell^p$*

$$f(x) = \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}$$

*where the series convergence is $b^p$-convergence.*

# Outline of the proof

## Definition (uniformly finite intersection)

A set family $\{U_i\}$ is called uniformly finite intersection with bound $N$, if there exists $N$ such that $\#\{i \in \mathbb{N}; x \in U_i\} \leq N$, for any $x \in \Omega$.

## Lemma (covering lemma )

*Let $0 < \delta < \frac{1}{4}$. We can choose $N$ (independ of $\delta$), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$ for $\Omega$.*

- *$E_i$ is measurable set for any $i \in \mathbb{N}$;*
- *$E_i \subset B(\lambda_i, \delta r(\lambda_i))$ for any $i \in \mathbb{N}$;*
- *$\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound $N$*

# Outline of the proof

## Lemma (bounded test lemma)

$$I_s f(x) := \int_\Omega \frac{r(y)^s}{d(x,y)^{n+s}} f(y)dy$$

*If* $s = 0$, *then* $I_s : L^p \to L^p$ *is bounded for* $p > 1$.
*If* $s > 0$, *then* $I_s : L^p \to L^p$ *is bounded for* $p \geq 1$.

# Outline of the proof

We put $0 < \delta < \frac{1}{4}$ (fixed later), $\{\lambda_i\} \subset \Omega$ and $\{E_i\}$ satisfying covering lemma. We consider the following operators;

$$A_{p,\{\lambda_i\}}(\{a_i\})(x) := \sum_{i=1}^{\infty} a_i R(x,\lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n} \text{ in } b^p$$

$$S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x,\lambda_i) f(\lambda_i) |E_i| \text{ in } b^p$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i$$

**Find a condition that $A_{p,\{\lambda_i\}} : \ell^p \to b^p(\Omega)$ is onto!**

# Outline of the proof

We check the following properties.

- $A_{p,\{\lambda_i\}} \circ U_{p,\{\lambda_i\}} = S_{p,\{\lambda_i\}}$
- $S_{p,\{\lambda_i\}} : b^p \to b^p$, $U_{p,\{\lambda_i\}} : b^p \to \ell^p$, $A_{p,\{\lambda_i\}} : \ell^p \to b^p$ are bounded operators.
- For enough small $\delta > 0$, $\|S_{p,\{\lambda_i\}} - id\| < 1$.

Hence $S_{p,\{\lambda_i\}} : b^p \to b^p$ is bijective.

# Definition and problem for Toeplitz operator

We consider the positive Toeplitz operator on $b^2$.

## Definition (Toeplitz operator)

We call the operator $T_\mu$ on $b^2$ the Toeplitz operator with symbol $\mu$, if

$$T_\mu := \int_\Omega R(x,y)f(y)d\mu(y).$$

**Problem.**
Describe conditions that the Toeplitz operator $T_\mu$ is **good** operator (for example bonded or compact).

# Definition of associate functions

## Definition (averaging function, Berezin transform)

For any $0 < \delta < 1$ and $1 < p < \infty$, we define

$$\hat{\mu}_\delta(x) := \frac{|\mu(E_\delta(x))|}{V(E_\delta(x))}$$

and

$$\tilde{\mu}_p(x) := \frac{\int_\Omega |R(x,y)|^p d\mu(y)}{\int_\Omega |R(x,y)|^p dy}$$

for any $x \in \Omega$.

# The preceding result for Toeplitz operator

**Theorem (B. R. Choe, Y. J. Lee and K. Na 2004)**

*Let* $1 \leq \sigma < \infty$ *and* $0 < \delta < 1$. *For* $\mu \geq 0$, *the following conditions are equivalent;*

1. $T_\mu \in S_\sigma$,
2. $\tilde{\mu}_2 \in L^\sigma(\lambda)$,
3. $\hat{\mu}_\delta \in L^\sigma(\lambda)$,
4. $\sum_j \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$.

*for some* $\{\lambda_j\}$ *satisfied with covering lemma.*

# Result 2

**Theorem (K. Tanaka (to appear in Osaka Journal))**

*Let* $\sigma > \frac{2(n-1)}{n+2}$ *and* $\mu \geq 0$*. Choose a constant* $\delta > 0$ *and a sequence* $\{\lambda_j\}$ *satisfying the conditions obtained by covering lemma. If* $\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$*, then* $T_\mu \in S_\sigma$*.*