A representation for harmonic Bergman functions and its applications

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Outline

1. Introduction
2. Modified harmonic Bergman kernel
3. Application
Harmonic Bergman Space $b^p$

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ is smooth bounded domain. $b^p(\Omega) := \{ f : \text{harmonic in } \Omega \text{ and } \| f \|_p < \infty \}$ $b^p$ is called harmonic Bergman space.

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- For any $x \in \Omega$, $f \in b^p(\Omega)$ has the following representation;

$$f(x) = \int_{\Omega} R(x, y)f(y)dy$$

$R(\cdot, \cdot)$ is called harmonic Bergman kernel.
Example of the harmonic Bergman kernel

The case $\Omega = \mathbb{B}$ (unit ball)

$$R_B(x, y) = \frac{(n - 4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x \cdot y + |x|^2|y|^2)^{1+\frac{n}{2}}}$$

$$= \frac{(n - 4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)((1 - |x|^2)(1 - |y|^2) + |x - y|^2)^{1+\frac{n}{2}}}$$
The recently result

**Theorem (H. Kang and H. Koo 2002)**

Let $\Omega$ be a smooth bounded domain and $\alpha$ and $\beta$ be multi-indices. Then, there exist $C_{\alpha,\beta} > 0$ and $C > 0$ such that for any $x, y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x, y)| \leq \frac{C_{\alpha,\beta}}{d(x, y)^{n+|\alpha|+|\beta|}}$$

$$R(x, x) \geq \frac{C}{r(x)^n}$$

where $d(x, y) := r(x) + r(y) + |x - y|$ and $r(x)$ is the distance between $x$ and boundary $\Omega$. 
We denote the **harmonic Bergman projection** by

\[ Pf(x) := \int_{\Omega} R(x, y)f(y)dy \quad f \in L^p(\Omega) \]

If \(1 < p < \infty\), then \(P : L^p(\Omega) \rightarrow b^p(\Omega)\) is bounded linear operator.
Theorem (K. Tanaka (to appear in Hiroshima Journal))

Let $1 < p < \infty$ and $\Omega$ be a smooth bounded domain. Then, we can choose $\{\lambda_i\} \subset \Omega$ such that for any $f \in b^p(\Omega)$ there exists $\{a_i\} \in \ell^p$ such that

$$f(x) = \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{\left(1-\frac{1}{p}\right)n}$$

where the convergence of series is $b^p$-convergence.
We choose a defining function $\eta$ for $\Omega$ such that $|\nabla \eta|^2 = 1 + \eta \omega$ for some $\omega \in C^\infty(\bar{\Omega})$. We denote the differential operator $K_1$ by

$$K_1 g := g - \frac{1}{2} \Delta (\eta^2 g),$$

and we denote the following kernel and projection;

$$R_1(x, y) := K_1 (R_x)(y) : \text{modified harmonic Bergman kernel},$$

$$P_1 f(x) := \int_{\Omega} R_1(x, y)f(y)dy : \text{modified projection.}$$
Some property

<table>
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<th>Theorem (B. R. Choe, H. Koo and H. Yi 2004)</th>
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<tr>
<td>1. ( P_1 f = f ) for any ( f \in b^1(\Omega) ).</td>
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<td>2. ( P_1 : L^p(\Omega) \rightarrow b^p(\Omega) ) is bounded for any ( 1 \leq p &lt; \infty ).</td>
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<td>3. For any multi-index ( \alpha ), there exists ( C_\alpha &gt; 0 ) such that</td>
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\[
|D^\alpha_x R_1(x, y)| \leq \frac{C_\alpha r(y)}{d(x, y)^{n+1+|\alpha|}}
\]

\[
|D^\alpha_y R_1(x, y)| \leq \frac{C_\alpha}{d(x, y)^{n+1}}
\]
Result 1

**Theorem (K. Tanaka (to appear in Osaka Journal))**

Let $1 \leq p < \infty$ and $\Omega$ be a smooth bounded domain. Then, we can choose $\{\lambda_i\} \subset \Omega$ such that for any $f \in b^p(\Omega)$ there exist $\{a_i\} \in \ell^p$

$$f(x) = \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}$$

where the series convergence is $b^p$-convergence.
Outline of the proof

Definition (uniformly finite intersection)
A set family \( \{U_i\} \) is called uniformly finite intersection with bound \( N \), if there exists \( N \) such that \( \#\{i \in \mathbb{N}; x \in U_i\} \leq N \), for any \( x \in \Omega \).

Lemma (covering lemma)
Let \( 0 < \delta < \frac{1}{4} \). We can choose \( N \) (independ of \( \delta \)), \( \{\lambda_i\} \subset \Omega \) and disjoint covering \( \{E_i\} \) for \( \Omega \).

- \( E_i \) is measurable set for any \( i \in \mathbb{N} \);
- \( E_i \subset B(\lambda_i, \delta r(\lambda_i)) \) for any \( i \in \mathbb{N} \);
- \( \{B(\lambda_i, 3\delta r(\lambda_i))\} \) is uniformly finite intersection with bound \( N \).
Lemma (bounded test lemma)

\[ I_s f(x) := \int_\Omega \frac{r(y)^s}{d(x, y)^{n+s}} f(y) dy \]

If \( s = 0 \), then \( I_s : L^p \to L^p \) is bounded for \( p > 1 \).
If \( s > 0 \), then \( I_s : L^p \to L^p \) is bounded for \( p \geq 1 \).
Outline of the proof

We put $0 < \delta < \frac{1}{4}$ (fixed later), $\{\lambda_i\} \subset \Omega$ and $\{E_i\}$ satisfying covering lemma. We consider the following operators:

$$A_{p,\{\lambda_i\}}(\{a_i\})(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n} \text{ in } b^p$$

$$S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i| \text{ in } b^p$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i$$

Find a condition that $A_{p,\{\lambda_i\}} : \ell^p \rightarrow b^p(\Omega)$ is onto!
Outline of the proof

We check the following properties.

- $A_{p,\{\lambda_i\}} \circ U_{p,\{\lambda_i\}} = S_{p,\{\lambda_i\}}$
- $S_{p,\{\lambda_i\}} : b^p \to b^p$, $U_{p,\{\lambda_i\}} : b^p \to \ell^p$, $A_{p,\{\lambda_i\}} : \ell^p \to b^p$ are bounded operators.
- For enough small $\delta > 0$, $\| S_{p,\{\lambda_i\}} - id \| < 1$.

Hence $S_{p,\{\lambda_i\}} : b^p \to b^p$ is bijective.
We consider the positive Toeplitz operator on $b^2$.

**Definition (Toeplitz operator)**

We call the operator $T_\mu$ on $b^2$ the Toeplitz operator with symbol $\mu$, if

$$T_\mu := \int_{\Omega} R(x, y)f(y)\,d\mu(y).$$

**Problem.**

Describe conditions that the Toeplitz operator $T_\mu$ is **good** operator (for example bonded or compact).
Definition of associate functions

Definition (averaging function, Berezin transform)

For any $0 < \delta < 1$ and $1 < p < \infty$, we define

$$\hat{\mu}_\delta(x) := \frac{|\mu(E_\delta(x))|}{V(E_\delta(x))}$$

and

$$\tilde{\mu}_p(x) := \frac{\int_\Omega |R(x, y)|^p d\mu(y)}{\int_\Omega |R(x, y)|^p dy}$$

for any $x \in \Omega$. 

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representation and application
The preceding result for Toeplitz operator

**Theorem (B. R. Choe, Y. J. Lee and K. Na 2004)**

Let $1 \leq \sigma < \infty$ and $0 < \delta < 1$. For $\mu \geq 0$, the following conditions are equivalent:

1. $T_\mu \in S_\sigma$,
2. $\tilde{\mu}_{2} \in L^\sigma(\lambda)$,
3. $\hat{\mu}_\delta \in L^\sigma(\lambda)$,
4. $\sum_j \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$.

for some $\{\lambda_j\}$ satisfied with covering lemma.
Result 2

Theorem (K. Tanaka (to appear in Osaka Journal))

Let $\sigma > \frac{2(n-1)}{n+2}$ and $\mu \geq 0$. Choose a constant $\delta > 0$ and a sequence $\{\lambda_j\}$ satisfying the conditions obtained by covering lemma. If $\sum_{j=1}^{\infty} \hat{\mu}_{\delta}(\lambda_j)^\sigma < \infty$, then $T_\mu \in S_\sigma$. 