Toeplitz operators on harmonic Bergman spaces

Kiyoki Tanaka

Osaka City University Advanced Mathematical Institute (OCAMI)

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Outline

1. Introduction

2. Harmonic Bergman space on the unit ball

3. Harmonic Bergman space on smooth bounded domain

4. Modified harmonic Bergman kernel

5. Application for Toeplitz operator
In 1922, S. Bergman suggested the following space (Bergman space):

\[ L^2_a(\mathbb{D}, dA) := \{ f : \mathbb{D} \to \mathbb{C} : f \text{ analytic in } \mathbb{D} \text{ and } \|f\|_2 < \infty \} \]

norm: \( \|f\|_2 := \left( \int_{\mathbb{D}} |f|^2 dA \right)^{1/2} \)
Bergman space on the unit disc

- $L^2_a(\mathbb{D}, dA) \subset L^2(\mathbb{D}, dA)$: closed subspace
- $L^2_a(\mathbb{D}, dA)$: reproducing kernel Hilbert space, i.e.,
  \[ \forall z \in \mathbb{D} \exists K(z, \cdot) \in L^2_a(\mathbb{D}, dA) \text{ s.t. } \forall f \in L^2_a(\mathbb{D}) \]
  \[ f(z) = \int_{\mathbb{D}} K(z, w)f(w)dA(w) \text{ (reproducing property)} \]
- $K(z, w) = \overline{K(w, z)}$: anti-symmetric
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  \[ K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2} \]
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Generalization

\[ L^2_a(\mathbb{D}, dA) := \{ f : \mathbb{D} \to \mathbb{C} : f : \text{analytic in } \mathbb{D} \text{ and } \| f \|_2 < \infty \}, \]

- exponent 2 → $p$.
- measure $A$ (norm $\| \cdot \|_2$) → weighted measure.
- analytic → solutions of differential equation.
- domain $\mathbb{D}$ → "general" domain.
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1 ≤ p < ∞, Ω ⊂ ℝ^n : domain.

\[ b^p(Ω) := \{ f : Ω → ℝ \text{ harmonic} \quad \text{and} \quad ∥f∥_p < ∞ \} : \text{harmonic Bergman space} \]

- \( b^p(Ω) \subset L^p(Ω) \): closed subspace
- In particular, \( b^2(Ω) \subset L^2(Ω) \): reproducing kernel Hilbert space
- \( f ∈ b^2(Ω) \) has a reproducing property: for \( x ∈ Ω \), there exists unique \( R(x, \cdot) ∈ b^2(Ω) \)

\[ f(x) = ∫_Ω R(x, y)f(y)dy. \]

\( R(\cdot, \cdot) : \text{harmonic Bergman kernel.} \)
$1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ : domain.

$b^p(\Omega) := \{f : \Omega \to \mathbb{R} \text{ harmonic and } \|f\|_p < \infty\}$ : harmonic Bergman space

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properties for harmonic Bergman kernel

- $R(x, y)$ has real value.
- Symmetric $R(x, y) = R(y, x)$.
- $\|R(x, \cdot)\|_{b^2} = R(x, x)$.
- If $\{e_m(\cdot)\}_{m \in \mathbb{N}}$ is orthogonal basis of $b^2(\Omega)$, then
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  R(x, y) = \sum_{m=1}^{\infty} e_m(x) e_m(y)
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We consider the orthogonal projection $P$ from $L^2(\Omega)$ to $b^2(\Omega)$. It is well-known that for $f \in L^2(\Omega)$, $P$ has the form

$$Pf(x) = \int_{\Omega} R(x, y)f(y)dy.$$ 

For a function $\varphi$, we denote Toeplitz operator $T_\varphi$ on $b^2(\Omega)$ by

$$T_\varphi f(x) := P(f\varphi)(x) = \int_{\Omega} R(x, y)f(y)\varphi(y)dy.$$ 

For a measure $\mu$, we denote Toeplitz operator $T_\mu$ on $b^2(\Omega)$ by

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When $\Omega = \mathbb{B}$ (unit ball),

$$R(x, y) = \frac{(n - 4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{n|\mathbb{B}|((1 - |x|^2)(1 - |y|^2) + |x - y|^2)^{1 + \frac{n}{2}}}$$

and

$$R_{\mathbb{B}}(x, x) = \frac{(n - 4)|x|^4 + 2n|x|^2 + n}{n|\mathbb{B}|(1 - |x|^2)^n}$$
When $\Omega = \mathbb{B}$: unit ball, $b^p(\mathbb{B}) \subset b^q(\mathbb{B}) \ (1 \leq q < p \leq \infty)$.

$b^p(\mathbb{B})^* \simeq b^q(\mathbb{B}) \ (1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1)$

$f \in b^p(\mathbb{B}) \ (1 \leq p \leq \infty)$ has the reproducing property

$$f(x) = \int_{\mathbb{B}} R(x, y)f(y)\,dy.$$ 

Harmonic Bergman projection $P$ is extended from $L^p(\mathbb{B}) \rightarrow b^p(\mathbb{B}) \ (1 < p < \infty)$ and $P : L^p(\mathbb{B}) \rightarrow b^p(\mathbb{B})$ is bounded.
harmonic Bergman space on the unit ball

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For $\alpha > -1$, we consider the weighted harmonic Bergman space $b^p_\alpha(\mathbb{B})$ denoted by

$$b^p_\alpha(\mathbb{B}) := \{ f : \text{harmonic on } \mathbb{B} \text{ and } \| f \|_{p,\alpha} < \infty \}$$

where

$$\| f \|_{p,\alpha} := \left( \int_{\mathbb{B}} |f(x)|^p dV_\alpha(x) \right)^{\frac{1}{p}}$$

and $dV_\alpha(x) = (1 - |x|^2)^\alpha dx$. 
Weighted kernel, projection

By same method, \( f \in b^2_\alpha(\mathbb{B}) \) has a reproducing property: for \( x \in \mathbb{B} \), there exists unique \( R_\alpha(x, \cdot) \in b^2_\alpha(\mathbb{B}) \)

\[
f(x) = \int_{\mathbb{B}} R_\alpha(x, y)f(y)dV_\alpha(y).
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Orthogonal projection \( P_\alpha \) form \( L^2(\mathbb{B}, dV_\alpha) \) to \( b^2_\alpha(\mathbb{B}) \) has the form

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Let $\alpha > 0$. Then,

- $b^p(\mathbb{B}) \subset b^p_{\alpha}(\mathbb{B})$.
- For $x, y \in \mathbb{B}$,
  \[
  R_{\alpha}(x, x) \approx \frac{1}{(1 - |x|)^{n+\alpha}}
  \]
  \[
  R_{\alpha}(x, y) \lesssim \frac{1}{|x - y|^{n+\alpha}}
  \]

- $P_{\alpha} : L^p(\mathbb{B}) \rightarrow b^p(\mathbb{B})$ is bounded for $1 \leq p$.
  \[
  P_{\alpha}f(x) = \int_{\mathbb{B}} f(y) R_{\alpha}(x, y)(1 - |y|^2)^{\alpha} \, dy.
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Properties for weighted harmonic Bergman kernel

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Known results of Toeplitz operators on $b^2(\mathbb{B})$

**Definition (averaging function, Berezin transform)**

For any $0 < \delta < 1$,

$$\hat{\varphi}_\delta(x) := \frac{\int_{E_\delta(x)} \varphi(y) \, dy}{V(E_\delta(x))} : \text{averaging function}$$

$$\tilde{\varphi}(x) := \frac{\int_{\mathbb{B}} |R(x, y)|^2 \varphi(y) \, dy}{R(x, x)} : \text{Berezin transform}$$

for any $x \in \mathbb{B}$, where $E_\delta(x) := \{y \in \mathbb{B} : |x - y| < \delta(1 - |x|)\}$.

We can describe the boundedness of Toeplitz operator $T_{\varphi}$ by using the above associate functions.
Known results of Toeplitz operators on $b^2(\mathbb{B})$

Theorem

Let $\varphi$ be a positive function on $\mathbb{B}$. Then, the following conditions are equivalent:

- $T_{\varphi}$ is bounded;
- averaging function $\hat{\varphi}$ is bounded function;
- Berezin transform $\tilde{\varphi}$ is bounded function.

Theorem

Let $\varphi$ be a positive function on $\mathbb{B}$. Then, the following conditions are equivalent:

- $T_{\varphi}$ is compact;
- averaging function $\hat{\varphi}(x) \to 0$ as $|x| \to 1$;
- Berezin transform $\tilde{\varphi}(x) \to 0$ as $|x| \to 1$. 
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We consider $\Omega$ is smooth bounded domain in $\mathbb{R}^n$. We have the following theorem.

**Theorem (Kang-Koo 2002)**

Let $\Omega$ be a smooth bounded domain and $\alpha$ and $\beta$ be multi-indices. Then, there exist $C_{\alpha,\beta} > 0$ and $C > 0$ such that for any $x, y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x, y)| \leq \frac{C_{\alpha,\beta}}{d(x, y)^{n+|\alpha|+|\beta|}}$$

and

$$R(x, x) \geq \frac{C}{r(x)^n}$$

where $d(x, y) := r(x) + r(y) + |x - y|$ and $r(x)$ is the distance between $x$ and boundary of $\Omega$. 
\[ 1 \leq p < \infty, \ f \in b^p(\Omega) \] has the reproducing property, that is

\[ f(x) = \int_{\Omega} R(x, y) f(y) \, dy \]

\[ Pf(x) := \int_{\Omega} R(x, y) f(y) \, dy \quad f \in L^p(\Omega) \]

**Harmonic Bergman projection**

\[ 1 < p < \infty \Rightarrow P : L^p(\Omega) \rightarrow b^p(\Omega) : \text{bounded linear operator} \]
Lemma (covering lemma)

Let $0 < \delta < \frac{1}{4}$. We can choose $N$ (independ of $\delta$), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$ for $\Omega$.

1. $E_i$ is measurable set for any $i \in \mathbb{N}$;
2. $E_i \subset B(\lambda_i, \delta r(\lambda_i))$ for any $i \in \mathbb{N}$;
3. $\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound $N$ where $r(x)$ denotes the distance between $x$ and $\partial \Omega$.
Theorem (T. 2012)

Let $1 < p < \infty$, $\Omega$ be a bounded smooth domain. There exists $0 < \delta_0$ such that if $\{\lambda_i\}$ satisfies covering lemma for $\delta < \delta_0$, then $A_{p,\{\lambda_i\}} : \ell^p \to b^p$ is a bounded onto map, where the operator $A_{p,\{\lambda_i\}}$ is defined by

$$A_{p,\{\lambda_i\}}\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where $r(x)$ denotes the distance between $x$ and $\partial \Omega$. 
Let denote $V : b^p(\Omega) \to l^p$ by

$$V_{p, \{\lambda_i\}} f := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}.$$

It is known that $A_{p, \{\lambda_i\}}^* = V_{q, \{\lambda_i\}}$ for $1 < p < \infty$, $q : \frac{1}{p} + \frac{1}{q} = 1$.

**Theorem (T. 2013)**

Let $1 < p < \infty$. We can choose a positive constant $\rho_0$ satisfying the following condition:

if $\{\lambda_i\}_i \subset \Omega$ satisfy quasi-hyperbolic distance $\rho(\lambda_i, \lambda_j) > \rho_0$ for $i \neq j$, then $V : b^p(\Omega) \to l^p$ is bounded and onto.

$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x, y}} \int_{\gamma} \frac{1}{r(z)} ds(z)$$
Interpolating of harmonic Bergman functions

Let denote $V : b^p(\Omega) \to l^p$ by

$$V_{p,\{\lambda_i\}} f := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}.$$

It is known that $A^*_{p,\{\lambda_i\}} = V_{q,\{\lambda_i\}}$ for $1 < p < \infty$, $q$: $\frac{1}{p} + \frac{1}{q} = 1$.

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$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x,y}} \int_{\gamma} \frac{1}{r(z)} \, ds(z)$$
Outline of the proof of representation theorem

We define the operators $U_{p,\{\lambda_i\}} : b^p \rightarrow \ell^p$ and $S_{p,\{\lambda_i\}} : b^p \rightarrow b^p$ as following;

\[
S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i|
\]

\[
U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i
\]

where $\{E_i\}_i$ is the disjoint covering of $\Omega$ such that $\lambda_i \in E_i$ for any $i \in \mathbb{N}$. Because $S = A \circ U$, by calculating $\|S - Id\|$, we can give the condition that $S$ is bijective.
Fix a defining function $\eta$ for $\Omega$ s.t.
$|\nabla \eta|^2 = 1 + \eta \omega$ for some $\omega \in C^\infty(\bar{\Omega})$.
We denote the differential operator $K_1$ by

$$
K_1 g := g - \frac{1}{2} \Delta(\eta^2 g)
$$

$$
R_1(x, y) := K_1(R_x)(y) \quad \text{: modified harmonic Bergman kernel},
$$

where $R_x(\cdot) := R(x, \cdot)$

$$
P_1 f(x) := \int_\Omega R_1(x, y) f(y) dy \quad \text{modified projection}.
$$
Some property

Theorem (Choe-Koo-Yi 2004)

- \( P_1 f = f \) for any \( f \in b^1(\Omega) \).
- \( P_1 : L^p(\Omega) \to b^p(\Omega) \) is bounded for any \( 1 \leq p < \infty \).
- For any multi-index \( \alpha \), there exists \( C_\alpha > 0 \) such that
  
  \[
  |D_x^\alpha R_1(x, y)| \leq \frac{C_\alpha r(y)}{d(x, y)^{n+1} + |\alpha|} \\
  |D_y^\alpha R_1(x, y)| \leq \frac{C_\alpha}{d(x, y)^{n+1}}
  \]
## Comparison between kernels

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Theorem (T. 2013)

Let $1 \leq p < \infty$ and $\Omega$ be a smooth bounded domain. Then, we can choose a sequence $\{\lambda_i\}$ in $\Omega$ such that $A_1 : \ell^p \to b^p$ is a bounded onto map, where the operator $A_1$ is defined by

$$A_1 \{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$
Harmonic Bloch space

\[ \mathcal{B} := \{ f : \Omega \to \mathbb{R} : \text{harmonic, } \| f \|_\mathcal{B} < \infty \} \]

\[ \| f \|_\mathcal{B} := \sup \{ r(x) |\nabla f(x)| : x \in \Omega \} \]

\[ (b^1)^* \cong \mathcal{B} \]

For fix \( x_0 \in \Omega , \)

\[ \mathcal{B}_0 := \{ f \in \mathcal{B} : f(x_0) = 0 \} \]
Theorem (T. 2013)

\( \Omega \) be a smooth bounded domain. Then, we can choose a sequence \( \{ \lambda_i \} \) in \( \Omega \) such that \( A_\infty : \ell^\infty \rightarrow \mathcal{B}_0 \) is a bounded onto map, where the operator \( A_\infty \) is defined by

\[
\tilde{f}(x) = \sum_{j=1}^{\infty} a_j \tilde{R}_1(x, \lambda_j) r(\lambda_j)^n,
\]

where \( \tilde{R}_1(x, y) = R_1(x, y) - R_1(0, y) \).
Definition and problem for Toeplitz operator

**Definition (Toeplitz operator)**

$T_\mu$ on $b^2$ the Toeplitz operator with symbol $\mu$

$$T_\mu f(x) := \int_\Omega R(x, y)f(y)d\mu(y).$$

**Problem.**

What condition is the Toeplitz operator $T_\mu$ good (bounded, compact and of Schatten $\sigma$-class $S^\sigma$ etc)?
Definition (Toeplitz operator)

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Averaging function, Berezin transform

**Definition (averaging function, Berezin transform)**

For any $0 < \delta < 1$ and $1 < p < \infty$,

\[
\hat{\mu}_\delta(x) := \frac{|\mu(E_\delta(x))|}{V(E_\delta(x))} : \text{averaging function}
\]

\[
\tilde{\mu}_p(x) := \frac{\int_\Omega |R(x, y)|^p d\mu(y)}{\int_\Omega |R(x, y)|^p dy} : \text{Berezin transform}
\]

for any $x \in \Omega$. 
The preceding result for Toeplitz operator

**Theorem (Choe-Lee-Na 2004)**

Let $1 \leq \sigma < \infty$ and $0 < \delta < 1$. For $\mu \geq 0$, the following conditions are equivalent;

- $T_\mu \in S_{\sigma}$,
- $\tilde{\mu}_2 \in L^\sigma(dV_R)$,
- $\hat{\mu}_\delta \in L^\sigma(dV_R)$,
- $\sum_j \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$.

for some $\{\lambda_j\}$ satisfied with covering lemma, where $dV_R = R(x, x)dx$.

c.f.

$T$ : compact operator on Hilbert space $\mathcal{H}$, $0 < \sigma \infty$

$T$ belongs to $\sigma$-Schatten class $S_{\sigma} \Leftrightarrow \sum_{m=1}^\infty s_m(T)^\sigma < \infty$

where $\{s_m(T)\}_m$ is singular value sequence of $T$. 
Extension of the previous theorem

**Theorem (T. 2013)**

Let $\sigma > \frac{2(n-1)}{n+2}$ and $\mu \geq 0$. Choose a constant $\delta > 0$ and a sequence \{\(\lambda_j\)\} satisfying the conditions obtained by covering lemma. If

$$\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^\sigma < \infty,$$

then $T_\mu \in S_\sigma$. 

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Outline of the proof

By the standard operator theory, $X \in S^\sigma$ and $Y$ is bdd operator $\Rightarrow XY, YX \in S^\sigma$.

- First, we can check the condition $A^* T_\mu A \in S^\sigma(\ell^2)$.
- $T_\mu = (US^{-1})^* A^* T_\mu A (US^{-1})$ belongs to $S^\sigma$. □
Outline of the proof

By the standard operator theory, 
\( X \in S^\sigma \) and \( Y \) is bdd operator \( \Rightarrow XY, YX \in S^\sigma \).

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- \( T_\mu = (US^{-1})^* \, A^* \, T_\mu \, A \, (US^{-1}) \) belongs to \( S^\sigma \). □
Outline of the proof

By the standard operator theory, \( X \in S^\sigma \) and \( Y \) is bdd operator \( \Rightarrow XY, YX \in S^\sigma \).

- First, we can check the condition \( A^* T_\mu A \in S^\sigma (\ell^2) \).
- \( T_\mu = (US^{-1})^* A^* T_\mu A (US^{-1}) \) belongs to \( S^\sigma \).  \( \square \)