Toeplitz operators on harmonic Bergman spaces

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Outline

Bergman space on the unit disc

In 1922, S. Bergman suggested the following space (Bergman space):

$$L_a^2(\mathbb{D}, dA) := \{f : \mathbb{D} \to \mathbb{C} : f : \text{analytic in } \mathbb{D} \text{ and } \|f\|_2 < \infty\}$$

$$\text{norm} : \|f\|_2 := \left(\int_{\mathbb{D}} |f|^2 dA\right)^{\frac{1}{2}}$$

- $L^2_a(\mathbb{D}, dA) \subset L^2(\mathbb{D}, dA)$: closed subspace
- $L^2_a(\mathbb{D}, dA)$: reproducing kernel Hilbert space, i.e., $\forall z \in \mathbb{D}\ \exists \overline{K(z,\cdot)} \in L^2_a(\mathbb{D}, dA)$ s.t. $\forall f \in L^2_a(\mathbb{D})$

$$f(z) = \int_{\mathbb{D}} K(z,w) f(w) dA(w) \text{ (reproducing property)}$$

- $K(z,w) = \overline{K(w,z)}$: anti-symmetric
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$$K(z,w) = \frac{1}{\pi(1 - z\bar{w})^2}$$

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Generalization

$$L^2_a(\mathbb{D}, dA) := \{f : \mathbb{D} \to \mathbb{C} : f : \text{analytic in } \mathbb{D} \text{ and } \|f\|_2 < \infty\},$$

- exponent $2 \to p$.
- measure A (norm $\|\cdot\|_2$) $\to$ weighted measure.
- analytic $\to$ solutions of differential equation.
- domain $\mathbb{D} \to$ "general" domain.

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$1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$: domain.
$b^p(\Omega) := \{f : \Omega \to \mathbb{R}$ harmonic and $\|f\|_p < \infty\}$: harmonic Bergman space

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- In particular, $b^2(\Omega) \subset L^2(\Omega)$: reproducing kernel Hilbert space
- $f \in b^2(\Omega)$ has a reproducing property: for $x \in \Omega$, there exists unique $R(x, \cdot) \in b^2(\Omega)$

$$f(x) = \int_\Omega R(x, y) f(y) dy.$$

$R(\cdot, \cdot)$: **harmonic Bergman kernel.**

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properties for harmonic Bergman kernel

- $R(x, y)$ has real value.
- symmetric $R(x, y) = R(y, x)$.
- $\|R(x, \cdot)\|_{b^2} = R(x, x)$.
- If $\{e_m(\cdot)\}_{m \in \mathbb{N}}$ is orthogonal basis of $b^2(\Omega)$, then

$$R(x, y) = \sum_{m=1}^{\infty} e_m(x) e_m(y)$$

for $x, y \in \Omega$.

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Projection, Toeplitz operator

We consider the orthogonal projection P from $L^2(\Omega)$ to $b^2(\Omega)$.
It is well-known that for $f \in L^2(\Omega)$, P has the form

$$Pf(x) = \int_\Omega R(x,y)f(y)dy.$$

For a function φ, we denote Toeplitz operator T_φ on $b^2(\Omega)$ by

$$T_\varphi f(x) := P(f\varphi)(x) = \int_\Omega R(x,y)f(y)\varphi(y)dy$$

For a measure μ, we denote Toeplitz operator T_μ on $b^2(\Omega)$ by

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harmonic Bergman kernel of unit ball

When $\Omega = \mathbb{B}$ (unit ball),

$$R(x, y) = \frac{(n-4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{n|\mathbb{B}|((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+\frac{n}{2}}}$$

and

$$R_{\mathbb{B}}(x, x) = \frac{(n-4)|x|^4 + 2n|x|^2 + n}{n|\mathbb{B}|(1-|x|^2)^n}$$

When $\Omega = \mathbb{B}$: unit ball, $b^p(\mathbb{B}) \subset b^q(\mathbb{B})$ $(1 \leq q < p \leq \infty)$.
$b^p(\mathbb{B})^* \simeq b^q(\mathbb{B})$ ($1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$)
$f \in b^p(\mathbb{B})$ $(1 \leq p \leq \infty)$ has the reproducing property

$$f(x) = \int_{\mathbb{B}} R(x, y) f(y) dy.$$

Harmonic Bergman projection P is extended from $L^p(\mathbb{B}) \to b^p(\mathbb{B})$ $(1 < p < \infty)$ and $P : L^p(\mathbb{B}) \to b^p(\mathbb{B})$ is bounded.

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Weighted harmonic Bergman space

For $\alpha > -1$, we consider the weighted harmonic Bergman space $b^p_\alpha(\mathbb{B})$ denoted by

$$b^p_\alpha(\mathbb{B}) := \{f : \text{ harmonic on } \mathbb{B} \text{ and } \|f\|_{p,\alpha} < \infty\}$$

where

$$\|f\|_{p,\alpha} := \left(\int_{\mathbb{B}} |f(x)|^p dV_\alpha(x)\right)^{\frac{1}{p}}$$

and $dV_\alpha(x) = (1 - |x|^2)^\alpha dx$.

Weighted kernel, projection

By same method, $f \in b^2_\alpha(\mathbb{B})$ has a reproducing property: for $x \in \mathbb{B}$, there exists unique $R_\alpha(x, \cdot) \in b^2_\alpha(\mathbb{B})$

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Orthogonal projection P_α form $L^2(\mathbb{B}, dV_\alpha)$ to $b^2_\alpha(\mathbb{B})$ has the form

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Properties for weighted harmonic Bergman kernel

Let $\alpha > 0$. Then,

- $b^p(\mathbb{B}) \subset b^p_\alpha(\mathbb{B})$.
- For $x, y \in \mathbb{B}$,

$$R_\alpha(x,x) \approx \frac{1}{(1-|x|)^{n+\alpha}}$$

$$R_\alpha(x,y) \lesssim \frac{1}{|x-y|^{n+\alpha}}$$

- $P_\alpha : L^p(\mathbb{B}) \to b^p(\mathbb{B})$ is bounded for $1 \leq p$.

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Known results of Toeplitz operators on $b^2(\mathbb{B})$

Definition (averaging function, Berezin transform)

For any $0 < \delta < 1$,

$$\hat{\varphi}_\delta(x) := \frac{\int_{E_\delta(x)} \varphi(y)dy}{V(E_\delta(x))} : \text{averaging function}$$

$$\tilde{\varphi}(x) := \frac{\int_{\mathbb{B}} |R(x,y)|^2 \varphi(y)dy}{R(x,x)} : \textbf{Berezin transform}$$

for any $x \in \mathbb{B}$, where $E_\delta(x) := \{y \in \mathbb{B} : |x - y| < \delta(1 - |x|)\}$.

We can describe the boundedness of Toeplitz operator T_φ by using the above associate functions.

Known results of Toeplitz operators on $b^2(\mathbb{B})$

Theorem

Let φ be a positive function on $\mathbb{B}$. Then, the following conditions are equivalent:

- T_φ *is bounded;*
- *averaging function* $\hat{\varphi}$ *is bounded function;*
- *Berezin transform* $\tilde{\varphi}$ *is bounded function.*

Theorem

Let φ be a positive function on $\mathbb{B}$. Then, the following conditions are equivalent:

- T_φ *is compact;*
- *averaging function* $\hat{\varphi}(x) \to 0$ *as* $|x| \to 1$;
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The preceding result

We consider Ω **is smooth bounded domain** in $\mathbb{R}^n$. We have the following theorem.

Theorem (Kang-Koo 2002)

Let Ω be a smooth bounded domain and α and β be multi-indices. Then, there exist $C_{\alpha,\beta} > 0$ and $C > 0$ such that for any $x, y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x,y)| \leq \frac{C_{\alpha,\beta}}{d(x,y)^{n+|\alpha|+|\beta|}}$$

and

$$R(x,x) \geq \frac{C}{r(x)^n}$$

where $d(x,y) := r(x) + r(y) + |x-y|$ and $r(x)$ is the distance between x and boundary of Ω.

$1 \leq p < \infty$, $f \in b^p(\Omega)$ has the reproducing property, that is

$$f(x) = \int_\Omega R(x,y)f(y)dy$$

$$Pf(x) := \int_\Omega R(x,y)f(y)dy \quad f \in L^p(\Omega)$$

harmonic Bergman projection
$1 < p < \infty \Rightarrow P : L^p(\Omega) \to b^p(\Omega)$: bounded linear operator

Preparation for results

Lemma (covering lemma)

Let $0 < \delta < \frac{1}{4}$. We can choose N (independ of δ), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$ for Ω.

1. *E_i is measurable set for any $i \in \mathbb{N}$;*
2. *$E_i \subset B(\lambda_i, \delta r(\lambda_i))$ for any $i \in \mathbb{N}$;*
3. *$\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound N*

where $r(x)$ denotes the distance between x and $\partial\Omega$.

Representation theorem

Theorem (T. 2012)

Let $1 < p < \infty$, Ω *be a bounded smooth domain. There exists* $0 < \delta_0$ *such that if* $\{\lambda_i\}$ *satisfies covering lemma for* $\delta < \delta_0$, *then* $A_{p,\{\lambda_i\}} : \ell^p \to b^p$ *is a bounded onto map, where the operator* $A_{p,\{\lambda_i\}}$ *is defined by*

$$A_{p,\{\lambda_i\}}\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where $r(x)$ *denotes the distance between* x *and* $\partial\Omega$.

Interpolating of harmonic Bergman functions

Let denote $V : b^p(\Omega) \to l^p$ by

$$V_{p,\{\lambda_i\}} f := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}.$$

It is known that $A^*_{p,\{\lambda_i\}} = V_{q,\{\lambda_i\}}$ for $1 < p < \infty$, q: $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem (T. 2013)

Let $1 < p < \infty$. We can choose a positive constant ρ_0 satisfying the following condition;
if $\{\lambda_i\}_i \subset \Omega$ satisfy quasi-hyperbolic distance $\rho(\lambda_i, \lambda_j) > \rho_0$ for $i \neq j$, then $V : b^p(\Omega) \to l^p$ is bounded and onto.

$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x,y}} \int_\gamma \frac{1}{r(z)} ds(z)$$

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Outline of the proof of representation theorem

We define the operators $U_{p,\{\lambda_i\}} : b^p \to \ell^p$ and $S_{p,\{\lambda_i\}} : b^p \to b^p$ as following;

$$S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i)|E_i|$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i$$

where $\{E_i\}_i$ is the disjoint covering of Ω such that $\lambda_i \in E_i$ for any $i \in \mathbb{N}$. Because $S = A \circ U$, by calculating $\|S - Id\|$, we can give the condition that S is bijective.

Modified harmonic Bergman kernel

Fix a defining function η for Ω s.t.
$|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$.
We denote the differential operator K_1 by

$$K_1 g := g - \frac{1}{2}\Delta(\eta^2 g)$$

$$R_1(x, y) := K_1(R_x)(y) \quad \text{: modified harmonic Bergman kernel,}$$

where $R_x(\cdot) := R(x, \cdot)$

$$P_1 f(x) := \int_\Omega R_1(x, y) f(y) dy \quad \text{modified projection.}$$

Some property

Theorem (Choe-Koo-Yi 2004)

- $P_1 f = f$ *for any* $f \in b^1(\Omega)$.
- $P_1 : L^p(\Omega) \to b^p(\Omega)$ *is bounded for any* $1 \leq p < \infty$.
- *For any multi-index* α*, there exists* $C_\alpha > 0$ *such that*

$$|D_x^\alpha R_1(x,y)| \leq \frac{C_\alpha r(y)}{d(x,y)^{n+1+|\alpha|}}$$

$$|D_y^\alpha R_1(x,y)| \leq \frac{C_\alpha}{d(x,y)^{n+1}}$$

Comparison between kernels

properties	Bergman kernel	modified kernel
symmetric	symmetric	non-symmetric
reproducing property	exist for $p \geq 1$	exist for $p \geq 1$
lower bdd	exist	not exist
upper bdd	$\frac{1}{d(x,y)^n}$	$\frac{r(y)}{d(x,y)^{n+1}}$
projection	bdd for $p > 1$	bdd for $p \geq 1$

Modified representation

Theorem (T. 2013)

Let $1 \leq p < \infty$ and Ω be a smooth bounded domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A_1 : \ell^p \to b^p$ is a bounded onto map, where the operator A_1 is defined by

$$A_1\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

Harmonic Bloch space

$$\mathcal{B} := \{f : \Omega \to \mathbb{R} : \text{ harmonic, } \|f\|_{\mathcal{B}} < \infty\}$$

$$\|f\|_{\mathcal{B}} := \sup\{r(x)|\nabla f(x)| : x \in \Omega\}$$

$$(b^1)^* \cong \mathcal{B}$$

For fix $x_0 \in \Omega$,

$$\mathcal{B}_0 := \{f \in \mathcal{B} : f(x_0) = 0\}$$

Representation for harmonic Bloch function

Theorem (T. 2013)

Ω be a smooth bounded domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A_\infty : \ell^\infty \to \mathcal{B}_0$ is a bounded onto map, where the operator A_∞ is defined by

$$f(x) = \sum_{j=1}^{\infty} a_j \tilde{R}_1(x, \lambda_j) r(\lambda_j)^n,$$

where $\tilde{R}_1(x, y) = R_1(x, y) - R_1(0, y)$.

Definition and problem for Toeplitz operator

Definition (Toeplitz operator)

T_μ on b^2 the Toeplitz operator with symbol μ

$$T_\mu f(x) := \int_\Omega R(x,y) f(y) d\mu(y).$$

Problem.
What condition is the Toeplitz operator T_μ **good** (bounded, compact and of Schatten σ-class S^σ etc) ?

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Averaging function, Berezin transform

Definition (averaging function, Berezin transform)

For any $0 < \delta < 1$ and $1 < p < \infty$,

$$\hat{\mu}_\delta(x) := \frac{|\mu(E_\delta(x))|}{V(E_\delta(x))} \text{: averaging function}$$

$$\tilde{\mu}_p(x) := \frac{\int_\Omega |R(x,y)|^p d\mu(y)}{\int_\Omega |R(x,y)|^p dy} \text{ : Berezin transform}$$

for any $x \in \Omega$.

The preceding result for Toeplitz operator

Theorem (Choe-Lee-Na 2004)

Let $1 \leq \sigma < \infty$ and $0 < \delta < 1$. For $\mu \geq 0$, the following conditions are equivalent;

- $T_\mu \in S_\sigma$,
- $\tilde{\mu}_2 \in L^\sigma(dV_R)$,
- $\hat{\mu}_\delta \in L^\sigma(dV_R)$,
- $\sum_j \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$.

for some $\{\lambda_j\}$ satisfied with covering lemma, where $dV_R = R(x,x)dx$.

c.f.

T : compact operator on Hilbert space $\mathcal{H}$, $0 < \sigma\infty$

T belongs to σ-Schatten class $S_\sigma \Leftrightarrow \sum_{m=1}^{\infty} s_m(T)^\sigma < \infty$

where $\{s_m(T)\}_m$ is singular value sequence of T.

Extension of the previous theorem

Theorem (T. 2013)

Let $\sigma > \frac{2(n-1)}{n+2}$ *and* $\mu \geq 0$. *Choose a constant* $\delta > 0$ *and a sequence* $\{\lambda_j\}$ *satisfying the conditions obtained by covering lemma. If* $\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$, *then* $T_\mu \in S_\sigma$.

Outline of the proof

By the standard operator theory,
$X \in S^\sigma$ and Y is bdd operator $\Rightarrow XY, YX \in S^\sigma$.

- First, we can check the condition $A^* T_\mu A \in S^\sigma(\ell^2)$.
- $T_\mu = (US^{-1})^* A^* T_\mu A(US^{-1})$ belongs to S^σ. $\square$

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