Interpolations of harmonic Bergman functions on smooth bounded domains

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Outline

1. Setting
2. Results
3. Outline of the proof

Bergman space (origin)

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc.

$$L_a^2(\mathbb{D}, dA) = \{f : \text{ analytic on } \mathbb{D}, \|f\|_2 < \infty\}$$

$f \in L_a^2(\mathbb{D}, dA)$ has the following formula:

$$f(z) = \int_{\mathbb{D}} f(w) \frac{1}{\pi(1 - z\bar{w})^2} dA(w) \text{ for } z \in \mathbb{D}$$

$K(z, w) = \frac{1}{\pi(1-z\bar{w})^2}$: Bergman kernel

For $f \in L_(^2\mathbb{D}, dA)$, $Pf(z) := \int_{\mathbb{D}} f(w) \frac{1}{\pi(1-z\bar{w})^2} dA(w)$: Bergman projection

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Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc.

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Examples of research

- Characterization of operators on $L^2_a(\mathbb{D})$ (Toeplitz operator, Hankel operator, composition operator, etc.)
- Change setting:
  - measure $dA \rightsquigarrow$ weighted measure $\varphi(w)dA(w)$ ($\varphi(w) = (1 - |w|^2)^\alpha$)
  - $\mathbb{D} \rightsquigarrow$ certain domain (upper-half plane, ball in $\mathbb{C}^n$, pseudo-convex domain)
  - analytic $\rightsquigarrow$ solution of certain differential equation (harmonic, heat equation)

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Harmonic Bergman space

$1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$: bounded smooth domain.
$b^p(\Omega) := \{f : \text{ harmonic in } \Omega \text{ and } \|f\|_p < \infty\}$
b^p : harmonic Bergman space.

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- $f \in b^2(\Omega)$ has a reproducing property:

$$f(x) = \int_\Omega R(x,y)f(y)dy \text{ for } x \in \Omega$$

$R(\cdot,\cdot)$: **harmonic Bergman kernel.**

Example of the harmonic Bergman kernel

When $\Omega = B$ (unit ball),

$$R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{n|B|((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+\frac{n}{2}}}$$

and

$$R_B(x,x) = \frac{(n-4)|x|^4 + 2n|x|^2 + n}{n|B|(1-|x|^2)^n}$$

Lemma (covering lemma)

Let $0 < \delta < \frac{1}{4}$. We can choose N (independ of δ), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$ for Ω.

1. *E_i is measurable set for any $i \in \mathbb{N}$;*
2. *$E_i \subset B(\lambda_i, \delta r(\lambda_i))$ for any $i \in \mathbb{N}$;*
3. *$\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound N*

Representation theorem

Theorem (T. (2012))

Let $1 < p < \infty$, Ω *be a bounded smooth domain. There exists* $0 < \delta_0$ *such that if* $\{\lambda_i\}$ *satisfies covering lemma for* $\delta < \delta_0$, *then* $A_{p,\{\lambda_i\}} : \ell^p \to b^p$ *is a bounded onto map, where the operator* $A_{p,\{\lambda_i\}}$ *is defined by*

$$A_{p,\{\lambda_i\}}\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where $r(x)$ *denotes the distance between* x *and* $\partial\Omega$.

Fix a defining function η for Ω s.t. $|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$. B.R. Choe, H. Koo and H. Yi (2004) introduced the following kernel

$$R_1(x, y) := R(x, y) - \frac{1}{2}\Delta_y(\eta^2(y)R_x(y)).$$

and shown that R_1 has the reproducing property.

Theorem (T. (2013))

Let $1 \le p < \infty$ and Ω be a smooth bounded domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A_1 : \ell^p \to b^p$ is a bounded onto map, where the operator A_1 is defined by

$$A_1\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

Interpolating of harmonic Bergman functions

Let denote $V : b^p(\Omega) \to l^p$ by

$$V_{p,\{\lambda_i\}} f := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}.$$

It is known that $A^*_{p,\{\lambda_i\}} = V_{q,\{\lambda_i\}}$ for $1 < p < \infty$, q: $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem (T. (2013))

Let $1 < p < \infty$. We can choose a positive constant ρ_0 satisfying the following condition;
if $\{\lambda_i\}_i \subset \Omega$ satisfy quasi-hyperbolic distance $\rho(\lambda_i, \lambda_j) > \rho_0$ for $i \neq j$, then $V : b^p(\Omega) \to l^p$ is bounded and onto.

$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x,y}} \int_\gamma \frac{1}{r(z)} ds(z)$$

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The preceding works

- R.R. Coifman - R. Rochberg (1980) : holomorphic on unit ball in $\mathbb{C}^n$
- B.R. Choe - H. Yi (1998) : harmonic on upper half space in $\mathbb{R}^n$
- K. Tanaka : harmonic on smooth bounded domain

The preceding result

Theorem (Kang-Koo 2002)

Let Ω be a smooth bounded domain and α and β be multi-indices. Then, there exist $C_{\alpha,\beta} > 0$ and $C > 0$ such that for any $x, y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x,y)| \leq \frac{C_{\alpha,\beta}}{d(x,y)^{n+|\alpha|+|\beta|}}$$

and

$$R(x,x) \geq \frac{C}{r(x)^n}$$

where $d(x,y) := r(x) + r(y) + |x-y|$ and $r(x)$ is the distance between x and boundary of Ω.

Outline of the proof of representation theorem

Lemma (covering lemma)

Let $0 < \delta < \frac{1}{4}$. We can choose N (independ of δ), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$ for Ω.

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Remark.
If there exists a positive constant $c > 0$ such that $\{B(\lambda_i, cr(\lambda_i))\}$ is uniformly finite intersection, then $A_{p,\{\lambda_i\}}$ and $V_{p,\{\lambda_i\}}$ are bounded for $1 < p < \infty$.

We define the operators $U_{p,\{\lambda_i\}} : b^p \to \ell^p$ and $S_{p,\{\lambda_i\}} : b^p \to b^p$ as following;

$$S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i|$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-\frac{1}{p})n}\}_i$$

where $\{E_i\}_i$ is the disjoint covering of Ω such that $\lambda_i \in E_i$ for any $i \in \mathbb{N}$. Because $S = A \circ U$, we may show that S is bijective map. By calculating $\|S - Id\|$, we can give the condition that S is bijective.

□

Interpolating of harmonic Bergman functions (again)

Let denote $V : b^p(\Omega) \to l^p$ by

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Theorem (T. (2013))

Let $1 < p < \infty$. We can choose a positive constant ρ_0 satisfying the following condition;
if $\{\lambda_i\}_i \subset \Omega$ satisfy quasi-hyperbolic distance $\rho(\lambda_i, \lambda_j) > \rho_0$ for $i \neq j$, then $V : b^p(\Omega) \to l^p$ is bounded and onto.

$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x,y}} \int_\gamma \frac{1}{r(z)} ds(z)$$

Outline of the proof of interpolating sequence theorem

We assume $\rho(\lambda_i, \lambda_j) > \rho > 1$.
Then, there exist $\delta > 0$ such that $\{B(\lambda_i, \delta r(\lambda_i))\}$ is disjoint. We put

$$W_{p,\{\lambda_i\}}\{a_i\} := V \circ A\{a_i\} = \{r(\lambda_j)^{\frac{n}{p}} \sum_{i=1}^{\infty} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

We choose the $\{\lambda_i\}_i$ such that the operator W is bijective. We put diagonal part of W

$$D\{a_i\} := \{a_j R(\lambda_j, \lambda_j) r(\lambda_j)^n\}$$

and off diagonal part of W

$$E\{a_i\} := \{r(\lambda_j)^{\frac{n}{p}} \sum_{i \neq j} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

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Because D : bijective and $W = D + E$, $\|E\| < \frac{1}{\|D^{-1}\|} \Rightarrow W$: bijective.
Moreover, since $R(x,x)r(x)^n \approx 1$, $\|D\| \approx 1$. We have only to give the estimate for $\|E\|$. By using Hölder's inequality, we have

$$\|E\{a_i\}\|_{l^p} \lesssim \delta^{\frac{-n(p-1)}{p}} \Big(\sum_{i=1}^{\infty} |a_i|^p r(\lambda_i)^{\frac{1}{q}} \sum_{j \neq i} r(\lambda_j)^{n-\frac{1}{q}} |R(\lambda_j, \lambda_i)| \Big)^{\frac{1}{p}}$$

We calculate a part of summation;

$$r(\lambda_i)^{\frac{1}{q}} \sum_{j \neq i} r(\lambda_j)^{n-\frac{1}{q}} |R(\lambda_j, \lambda_i)| \lesssim \int_{\Omega \setminus E_\delta(\lambda_i)} \frac{r(\lambda_i)^{\frac{1}{q}} r(z)^{-\frac{1}{q}}}{d(z, \lambda_i)^n} dz$$

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Lemma (Pseudo-hyperbolic distance)

There exist constants $C_1 > 0$ and $C_2 > 0$ such that for any $x, y \in \Omega$

$$\frac{1}{d(x,y)} \leq \frac{\exp(-\frac{\rho(x,y)-C_1}{C_2})}{\min\{r(x), r(y)\}}.$$

By using quasi-hyperbolic distance condition $\rho(\lambda_i, \lambda_j) > \rho$, for $0 < \epsilon < 1$ we have

$$\int_{\Omega \setminus E_\delta(\lambda_i)} \frac{r(\lambda_i)^{\frac{1}{q}} r(z)^{-\frac{1}{q}}}{d(z,\lambda_i)^n} dz \lesssim \exp(-\epsilon(\frac{\rho - C_1}{C_2})) \to 0$$

as $\rho \to \infty$.

Then, there exists $\rho_0 > 0$ such that $\|E\| < \frac{1}{\|D^{-1}\|}$ if $\rho(\lambda_i, \lambda_j) > \rho_0$ $(i \neq j)$.

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