Harmonic Bergman spaces on smooth bounded domain

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Contents.

1 harmonic Bergman space b^p

$1 \leq p < \infty$

$\Omega \subset \mathbb{R}^n$: smooth bounded domain (i.e. $\partial\Omega$ is C^∞)

$b^p(\Omega) := \{f$: harmonic on Ω and $\|f\|_p < \infty\}$

b^p is called the **harmonic Bergman space**.

(where $\|f\|_p := (\int_\Omega |f|^p dx)^{\frac{1}{p}}$)

Basic properties;

- $b^p(\Omega) \subset L^p(\Omega)$: closed subspace
- $\forall x \in \Omega$, $\forall f \in b^p(\Omega)$

$$f(x) = \int_\Omega R(x,y)f(y)dy$$

$R(\cdot,\cdot)$ is called **harmonic Bergman kernel**.

- Example of the harmonic Bergman kernel

The case $\Omega = \mathbb{B}$

$$R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x\cdot y + |x|^2|y|^2)^{1+\frac{n}{2}}}$$

$$= \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+\frac{n}{2}}}$$

Theorem A (H. Kang and H. Koo [6] 2002)

Ω: smooth bounded domain, α, β: multi-indices.

$\exists C_{\alpha,\beta}$ s.t. $\forall x$, $\forall y \in \Omega$

$$|D_x^\alpha D_y^\beta R(x,y)| \le \frac{C_{\alpha,\beta}}{d(x,y)^{n+|\alpha|+|\beta|}}$$

$\exists C$ s.t. $\forall x \in \Omega$

$$R(x,x) \ge \frac{C}{r(x)^n}$$

where $d(x,y) := r(x) + r(y) + |x-y|$

We denote the **harmonic Bergman projection** by

$$Pf(x) := \int_\Omega R(x,y)f(y)dy \quad f \in L^p(\Omega)$$

If $1 < p < \infty$, then $P : L^p(\Omega) \to b^p(\Omega)$: bounded linear operator.
We obtain dualities for $1 < p < \infty$,

$$b^p(\Omega)^* \cong b^q(\Omega)$$

where q is an exponential conjugate and we consider the pairing

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx.$$

2 atomic decomposition for the case $p > 1$

Theorem 1 . [9]

$1 < p < \infty$

Ω: smooth bounded domain.

$\exists \{\lambda_i\} \subset \Omega$ s.t. $\forall f \in b^p(\Omega)$ $\exists \{a_i\} \in l^p$

$$f(x) = \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}$$

where $r(x) := \mathrm{dist}(x, \partial\Omega)$ and the convergence of series is b^p-convergence.

Remark.

$\|R(\cdot, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\| \approx 1.$

3 modified harmonic Bergman kernel

η : defining function for Ω s.t. $|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$
We denote the differential operator K_1 by

$$K_1 g := g - \frac{1}{2}\Delta(\eta^2 g),$$

and we denote the following;

$$R_1(x,y) := K_1(R_x)(y) \quad \text{modified harmonic Bergman kernel}$$

$$P_1 f(x) := \int_\Omega R_1(x,y) f(y) dy$$

.

Theorem B (B. R. Choe, H. Koo and H. Yi [2] 2004)

- $\forall f \in b^1(\Omega) \quad P_1 f = f$
- $1 \le p < \infty \Rightarrow P_1 : L^p(\Omega) \to b^p(\Omega)$ is bounded.
- α: multi-indices $\quad \exists C_\alpha > 0$ s.t.

$$|D_x^\alpha R_1(x,y)| \le \frac{C_\alpha r(y)}{d(x,y)^{n+1+|\alpha|}}$$

$$|D_y R_1(x,y)| \le \frac{C}{d(x,y)^{n+1}}$$

Theorem 2.

$1 \leq p < \infty$

Ω: smooth bounded domain.

$\exists \{\lambda_i\} \subset \Omega$ s.t. $\forall f \in b^p(\Omega)$ $\exists \{a_i\} \in l^p$

$$f(x) = \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}$$

where the series convergence is b^p-convergence.

4 harmonic Bloch space

In this section, we assume $0 \in \Omega$.

Definition (harmonic Bloch space)

$\mathcal{B} := \{f : \text{ harmonic } \text{ and} \|f\|_{\mathcal{B}} < \infty\}$: harmonic Bloch space

$\tilde{\mathcal{B}} := \{f \in \mathcal{B} : f(0) = 0\}$

where $\|f\|_{\mathcal{B}} := \sup\{r(x)|\nabla f(x)|; x \in \Omega\}$ is called Bloch norm.

Properties

- $(b^1)^* \backsimeq \mathcal{B}$
- For $1 < \forall p < \infty$, $\mathcal{B} \subset b^p$
- $b^\infty \subset \mathcal{B}$: continious

Theorem 3.

Ω: smooth bounded domain.

$\exists\{\lambda_i\} \subset \Omega$ s.t. $\forall f \in \tilde{\mathcal{B}}$ $\exists\{a_i\} \in l^\infty$

$$f(x) = \sum_{i=1}^{\infty} a_i \tilde{R}_1(x, \lambda_i) r(\lambda_i)^n$$

where $\tilde{R}_1(x, y) := R_1(x, y) - R_1(0, y)$.

5 outline of the proof for atomic decompositions

We introduce the expression to describe the small interval of $\{\lambda_i\}$.

Definition (uniformly finite intersection)

A open set family $\{U_i\}$ is called uniformly finite intersection with bound N

$\overset{\text{def}}{\Leftrightarrow} \exists N$ s.t. $\forall x \in \Omega \ \#\{i \in \mathbb{N}; x \in U_i\} \leq N$

Lemma 1. (covering lemma)

Let $0 < \delta < \frac{1}{4}$.

We can choose N (independ of δ), $\{\lambda_i\} \subset \Omega$ and disjoint covering $\{E_i\}$.

(a)$\forall i \in \mathbb{N} \ E_i$ is measurable set

(b)$\forall i \in \mathbb{N} \ E_i \subset B(\lambda_i, \delta r(\lambda_i))$

(c)$\{B(\lambda_i, 3\delta r(\lambda_i))\}$ is uniformly finite intersection with bound N

Lemma 2

Let s, t be nonnegative real numbers. If $s+t>0$ and $t<1$, then there exists a constant C such that

$$\int_\Omega \frac{dy}{d(x,y)^{n+s} r(y)^t} \leq \frac{C}{r(x)^{s+t}}$$

for every $x \in \Omega$.

Lemma 3.

$$I_\alpha f(x) := \int_\Omega \frac{r(y)^\alpha}{d(x,y)^{n+\alpha}} f(y) dy$$

If $\alpha = 0$, then $I_\alpha : L^p \to L^p :$ bounded for $p>1$.
If $\alpha > 0$, then $I_\alpha : L^p \to L^p :$ bounded for $p \geq 1$.

proof of theorem 1

We put $0 < \delta < \frac{1}{4}$ (fixed later), $\{\lambda_i\} \subset \Omega$ and $\{E_i\}$ by lemma 1.
We consider the following operators;

$$A_{p,\{\lambda_i\}}(\{a_i\})(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{\frac{n}{q}} \text{ in } b^p$$

$$S_{p,\{\lambda_i\}} f(x) := \sum_{i=1}^{\infty} R(x, \lambda_i) f(\lambda_i) |E_i| \text{ in } b^p$$

$$U_{p,\{\lambda_i\}}(f) := \{|E_i| f(\lambda_i) r(\lambda_i)^{-\frac{n}{q}}\}_i$$

We change theorem 1.
Find a condition that $A_{p,\{\lambda_i\}} : \ell^p \to b^p(\Omega)$ is bounded!

- $A_{p,\{\lambda_i\}} \circ U_{p,\{\lambda_i\}} = S_{p,\{\lambda_i\}}$
- $S_{p,\{\lambda_i\}} : b^p \longrightarrow b^p$, $\quad U_{p,\{\lambda_i\}} : b^p \longrightarrow \ell^p$,
  $A_{p,\{\lambda_i\}} : \ell^p \longrightarrow b^p$ are bounded operators.
- For enough small $\delta > 0$, $\|S_{p,\{\lambda_i\}} - I\| < 1$.

Hence $S_{p,\{\lambda_i\}} : b^p \longrightarrow b^p$ is bijective.

The following is a part of calculation.

$$
\begin{aligned}
(I - S)f(x) &= \int_\Omega f(y)R(x,y)dy - \sum_{i=1}^{\infty} R(x,\lambda_i)f(\lambda_i)|E_i| \\
&= \sum_{i=1}^{\infty} \int_{E_i} f(y)(R(x,y) - R(x,\lambda_i))dy =: F_1(x) \\
&+ \sum_{i=1}^{\infty} \int_{E_i} (f(y) - f(\lambda_i))R(x,\lambda_i)dy =: F_2(x)
\end{aligned}
$$

$$
\begin{aligned}
|F_1(x)| &\lesssim \sum_{i=1}^{\infty} \int_{E_i} |f(y)||y-\lambda_i||\nabla_y R(x,\bar{y})|dy \\
&\lesssim \delta \sum_{i=1}^{\infty} \int_{E_i} |f(y)| r(\lambda_i) \frac{1}{d(x,\bar{y})^{n+1}} dy \\
&\lesssim \delta \sum_{i=1}^{\infty} \int_{E_i} \frac{r(y)}{d(x,y)^{n+1}} |f(y)| dy \\
&= \delta I_1 |f|(x).
\end{aligned}
$$

Hence, $\|F_1\|_p \lesssim \delta \|f\|_p$.
By similar calculation, we get $\|S_{p,\{\lambda_i\}} - I\| < C\delta$. $\square$

6 Interpolating sequence

Atomic decomposition problem is the existence of surjective map from ℓ^p to b^p.

In this section, we consider the map from b^p to ℓ^p.

We denote a operator $V_{p,\{\lambda_i\}} : b^p \longrightarrow \ell^p$ as follow;

$$Vf := \{f(\lambda_i)r(\lambda_i)^{\frac{n}{p}}\}$$

Definition (quasihyperbolic metric)

$$\rho(x, y) := \inf_{\gamma_{xy} \in \Gamma_{xy}} \int_{\gamma_{xy}} \frac{1}{r(z)} ds(z)$$

where

$\Gamma_{xy} := \{$all smooth curve such that initial point is x and final point is y$\}$

The following estimate is known;

$$\rho(x,y) \leq C_1 \log \frac{d(x,y)}{\min\{r(x), r(y)\}} + C_2$$

$$\frac{\min\{r(x), r(y)\}}{d(x,y)} \leq \exp\{-c'(\rho(x,y) - c'')\}$$

Theorem 4.

Let $1 < p < \infty$. There exists $\rho_0 > 0$ such that for any $\{\lambda_i\}$ with $\rho_0 < \rho(\lambda_i, \lambda_i)$ if $i \neq j$, $V_{p,\{\lambda_i\}} : b^p \to \ell^p$ is surjective.

outline of the proof

We assume that $\delta > 0$ is fixed enough small and $\{B(\lambda_i, \delta r(\lambda_i))\}_i$ is separated. We define $W_{p,\{\lambda_i\}} : \ell^p \to \ell^p$, $D_{p,\{\lambda_i\}}$ and $E_{p,\{\lambda_i\}}$ as follow;

$$W_{p,\{\lambda_i\}}(\{a_i\}) := \{\sum_{j=1}^{\infty} a_j R(\lambda_i, \lambda_j) r(\lambda_j)^{\frac{n}{q}} r(\lambda_i)^{\frac{n}{p}}\}_i$$

$$D_{p,\{\lambda_i\}}(\{a_i\}) := \{a_i R(\lambda_i, \lambda_i) r(\lambda_i)^n\}_i$$

$$E_{p,\{\lambda_i\}}(\{a_i\}) := \{\sum_{j\neq i} a_j R(\lambda_i, \lambda_j) r(\lambda_j)^{\frac{n}{q}} r(\lambda_i)^{\frac{n}{p}}\}_i$$

By standard argument, we should only show $\|E\| < \|D\|$.

We obtain $C \leq \|D\|$ by the behavior of the kernel.
Hence, we calculate $\|E\{a_i\}\|_{\ell^p}$. The trouble calculation's result is the following estimate.

$$\|E\{a_i\}\|_{\ell^p}^p \lesssim \sum_{i=1}^{\infty} |a_i|^p r(\lambda_i)^{\frac{1}{q}} \int_{\cup_{j\neq i} B(\lambda_j, \delta r(\lambda_j))} \frac{dy}{d(\lambda_i, y)^n r(y)^{\frac{1}{q}}}.$$

We can control the right side by using the quasihyperbolic metric.

7 application of atomic decomposition

We consider only Hilbert space b^2.

Definition (Toeplitz operator)

For $\mu \in M(\Omega)$, the Toeplitz operator T_μ with symbol μ is defined by

$$T_\mu f(x) := \int_\Omega R(x,y) f(y) d\mu(y) \quad (x \in \Omega). \tag{1}$$

Problem

Describe the characters of Toeplitz operators!

Definition

Let $\mu \geq 0$. For $\delta \in (0,1)$, the averaging functon $\hat{\mu}_\delta$ is defined by

$$\hat{\mu}_\delta(x) := \frac{\mu(B(x, \delta r(x)))}{V(B(x, \delta r(x)))} \quad (x \in \Omega). \tag{2}$$

Also, for $1 < p < \infty$, we define the Berezin p-transform $\tilde{\mu}_p$ on Ω by

$$\tilde{\mu}_p(x) := \frac{\int_\Omega |R(x,y)|^p d\mu(y)}{\|R(x,\cdot)\|_{b^p}^p} \quad (x \in \Omega). \tag{3}$$

Definition (Schatten class operator) Let X be a separable Hilbert space and T be a compact operator. We say T is a Schatten p-class operator $(0 < p)$ if

$$\|T\|_{S_p(X)} := \Big(\sum_{m=1}^{\infty} |s_m(T)|^p \Big)^{\frac{1}{p}} < \infty$$

where $s_m(T)$ is the singular value sequence of T.

Fact for the Hilbert space $b^2(\Omega)$ [4]

Let $1 \leq p < \infty$ and $\delta \in (0,1)$. For $\mu \geq 0$, the following conditions are equivalent:

(a) $T_\mu \in S_p$

(b) $\tilde{\mu} \in L^p(\lambda)$

(c) $\hat{\mu}_\delta \in L^p(\lambda)$

(d) $\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^p < \infty$

where $d\lambda(x) := R(x,x)dx$.

Theorem 4.
Let $\frac{2(n-1)}{n+2} < p$. If $\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^p < \infty$, then $T_\mu \in S_p$.

proof
Lemma ([7])
If T is a compact operator on a Hilbert space H and $0 < p \leq 2$, then for any orthonormal basis $\{e_n\}$ we have

$$\|T\|_{S_p(X)}^p \leq \sum_n \sum_k |\langle Te_n, e_k \rangle|^p. \tag{4}$$

We use Luecking's ideas [7]. We should show the estimate for $\sum_i \sum_j |\langle A^* T_\mu A e_i, e_j \rangle|^p$, where $\{e_n\}$ is orthonormal basis for ℓ^2 and A is the operator of the atomic decomposition.

First, we calculate $\langle A^*T_\mu Ae_i, e_j\rangle$.

$$Ae_i(x) = R_1(x, \lambda_i)r(\lambda_i)^{\frac{n}{2}}$$

and

$$T_\mu Ae_i(x) = r(\lambda_i)^{\frac{n}{2}} \int_\Omega R_1(x, y)R_1(y, \lambda_i)d\mu(y).$$

Therefore, we have

$$\langle A^*T_\mu Ae_i, e_j\rangle = r(\lambda_i)^{\frac{n}{2}} r(\lambda_j)^{\frac{n}{2}} \int_\Omega R_1(y, \lambda_i)R_1(y, \lambda_j)d\mu(y).$$

And we have

$$\sum_i \sum_j |\langle A^* T_\mu A e_i, e_j\rangle|^p$$

$$= \sum_i \sum_j \left| r(\lambda_i)^{\frac{n}{2}} r(\lambda_j)^{\frac{n}{2}} \int_\Omega R_1(x, \lambda_i) R_1(x, \lambda_j) d\mu(x) \right|^p$$

$$\lesssim \sum_i \sum_j r(\lambda_i)^{\frac{np}{2}} r(\lambda_j)^{\frac{np}{2}} \Big(\sum_k \int_{B(\lambda_k, \delta r(\lambda_k))} \frac{r(\lambda_i)}{d(x, \lambda_i)^{n+1}} \frac{r(\lambda_j)}{d(x, \lambda_j)^{n+1}} d|\mu|(x) \Big)^p$$

$$\lesssim \sum_j r(\lambda_i)^{\frac{np}{2}} r(\lambda_j)^{\frac{np}{2}} \Big(\sum_k |\mu|(B(\lambda_k, \delta r(\lambda_k))) \frac{r(\lambda_i)}{d(\lambda_k, \lambda_i)^{n+1}} \frac{r(\lambda_j)}{d(\lambda_k, \lambda_j)^{n+1}} \Big)^p$$

$$\lesssim \sum_k \hat{\mu}_\delta(\lambda_k)^p r(\lambda_k)^{np} \Big(\sum_i \frac{r(\lambda_i)^{\frac{np}{2}+p}}{d(\lambda_k, \lambda_i)^{(n+1)p}} \Big)^2$$

$$= \sum_k \hat{\mu}_\delta(\lambda_k)^p \Big(\sum_i \frac{r(\lambda_k)^{\frac{np}{2}} r(\lambda_i)^{\frac{np}{2}+p}}{d(\lambda_k, \lambda_i)^{(n+1)p}} \Big)^2$$

So, we check the term of the sum for i.

$$\sum_i \frac{r(\lambda_k)^{\frac{np}{2}} r(\lambda_i)^{\frac{np}{2}+p}}{d(\lambda_k,\lambda_i)^{(n+1)p}} \lesssim \sum_i \int_{B(\lambda_i,\delta\lambda_i)} \frac{r(\lambda_k)^{\frac{np}{2}} r(\lambda_i)^{\frac{np}{2}+p-n}}{d(\lambda_k,\lambda_i)^{n+(n+1)p-n}} dy$$

$$\lesssim \int_\Omega \frac{r(\lambda_k)^{\frac{np}{2}} r(y)^{\frac{np}{2}+p-n}}{d(\lambda_k,y)^{n+(n+1)p-n}} dy \lesssim 1.$$

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