On the biharmonic Bergman kernel for the outside of the unit ball

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Setting

Let a dimension $N > 2$, $\mathbb{B}$ be the unit ball in $\mathbb{R}^N$ and $\mathbb{S} = \partial\mathbb{B}$.
For $D = \mathbb{B}, \mathbb{B} \setminus \{0\}$ or $\mathbb{R}^N \setminus \overline{\mathbb{B}}$, a positive integer m, $1 \leq p < \infty$, $\alpha > -N$ and $\beta > -1$, we define

$$b^{m,p}_{\alpha,\beta}(D) := H^m(D) \cap L^p(D, |x|^\alpha \left|1 - |x|^2\right|^\beta dx)$$

where $H^m(D) := \{u \in C^\infty(D) : \Delta^m u = 0\}$.
We call $b^{m,p}_{\alpha,\beta}(D)$ the (weighted) polyharmonic Bergman space.

Remark.

$$b^{1,p}_{0,0}(\mathbb{R}^N) = \{0\} \text{ and } b^{1,p}_{0,0}(\mathbb{R}^N \setminus \{0\}) = \{0\}.$$

Previous works

In particular, $b^{m,2}_{\alpha,\beta}(D)$ is the reproducing kernel Hilbert space with the reproducing kernel $R_{D,m,\alpha,\beta}(x,y)$, i.e., for $x \in D$ and $f \in b^{m,2}_{\alpha,\beta}(D)$,

$$f(x) = \int_D R_{D,m,\alpha,\beta}(x,y) f(y) |x|^\alpha \left| 1 - |x|^2 \right|^\beta dx.$$

Example.

$$R_{\mathbb{B},1,0,0}(x,y) = \frac{(N-4)|x|^4|y|^4 + (8x\cdot y - 2N - 4)|x|^2|y|^2 + N}{N|\mathbb{B}|(1 - 2x\cdot y + |x|^2|y|^2)^{\frac{N}{2}+1}}$$

- $R_{\mathbb{B},1,0,\beta}$: R. Coifman and R. Rochberg (1980)
- $R_{\mathbb{B},1,-4,0}$, $R_{\mathbb{R}^N\setminus\overline{\mathbb{B}},1,0,0}$: Z. G. Zhao (2014)
- $R_{\mathbb{B},2,0,0}$: T. (submitted, MSJ Autumn Meeting 2015)

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Today's target

Remark. We define by $\hat{R}_{\mathbb{B},1,0,0}(x,y)$ a harmonic extension of $R_{\mathbb{B},1,0,0}(x,y)$ for $x, y \in \{1 - 2x \cdot y + |x|^2|y|^2 \neq 0\}$. If $N \geq 4$,

$$\hat{R}_{\mathbb{B},1,0,0}(x,y) = R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},1,0,0}(x,y)$$

for $x, y \in \mathbb{R}^N \setminus \overline{\mathbb{B}}$.

First, we discuss the forms of $R_{\mathbb{B},1,\alpha,\beta}$ and $R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},1,\alpha,\beta}$.

Second, we discuss the forms of $R_{\mathbb{B},2,\alpha,\beta}$ and $R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},2,\alpha,\beta}$.

Theorem

Let $\beta \in \mathbb{N}_0$ and $N + \alpha > 0$. A domain of $R_{\mathbb{B},2,\alpha,\beta}(x,y)$ is naturally extended to $x, y \in \{1 - 2x \cdot y + |x|^2|y|^2 \neq 0\}$ (we write as $\hat{R}_{\mathbb{B},2,\alpha,\beta}$). Moreover, for $\alpha < N - 2\beta - 8$,

$$\hat{R}_{\mathbb{B},2,\alpha,\beta}(x,y) = (-1)^\beta R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},2,\alpha,\beta}(x,y)$$

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for $x,y \in \mathbb{R}^N \setminus \overline{\mathbb{B}}$.

Weighted harmonic Bergman space

For positive measure ν on $\mathbb{B}$, we put

$$b^2_\nu = H^1(\mathbb{B}) \cap L^2(\mathbb{B}, d\nu).$$

Lemma (M. Nishio and T.)

Let ν be a positive finite radial Borel measure (we write $d\nu = d\tilde{\nu} d\sigma$). Then, $\tilde{\nu}([r, 1)) > 0$ for any $r \in [0, 1)$ if and only if b^2_ν is a reproducing kernel Hilbert space. Moreover,

$$\left\{ \frac{e^k_j(x)}{\int_{[0,1)} r^{2k} d\tilde{\nu}(r)} \right\}_{j=1,\cdots,h_k, k=0,\cdots} \text{ is an orthonormal basis of } b^2_\nu$$

where $e^k_j(x)$ satisfies that $\{e^k_j|_{\mathbb{S}}\}$ is an orthonormal basis of the k-th homogenuous harmonic polynomials $\mathcal{H}^k(\mathbb{S}) \subset L^2(\mathbb{S}, d\sigma)$.

Harmonic Bergman kernel of $b^{1,2}_{\alpha,\beta}(\mathbb{B})$

Hence, if $N+\alpha>0$ and $\beta>-1$, then

$$\left\{\sqrt{\frac{2}{N|\mathbb{B}|B(k+\frac{N+\alpha}{2},\beta+1)}}e^k_j(x)\right\}_{j=1,\cdots,h_k,k=0,\cdots} : \text{O.N.B. of } b^{1,2}_{\alpha,\beta}(\mathbb{B}).$$

Therefore, we have

$$R_{\mathbb{B},1,\alpha,\beta}(x,y)=\frac{2}{N|\mathbb{B}|}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{N+\alpha}{2}+\beta+1)}{\Gamma(k+\frac{N+\alpha}{2})\Gamma(\beta+1)}Z_k(x,y)$$

where $Z_k(\theta,\eta)=\sum_{j=1}^{h_k}e^k_j(\theta)e^k_j(\eta)$: zonal harmonic.

In particular,

$$R_{\mathbb{B},1,\alpha,0}(x,y) = R_{\mathbb{B},1,0,0}(x,y) + \frac{\alpha}{N|\mathbb{B}|}P(x,y).$$

where $P(x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{\frac{N}{2}}}$: extended Poisson kernel.
Moreover, we have

$$\frac{d}{dt}\left(t^{\frac{N+\alpha}{2}+\beta+1}R_{\mathbb{B},1,\alpha,\beta}(tx,y)\right)\Big|_{t=1} = (\beta+1)R_{\mathbb{B},1,\alpha,\beta+1}(x,y)$$

for $x, y \in \mathbb{B}$.

$b^{1,2}_{\alpha,\beta}(\mathbb{B}\setminus\{0\})$ and $b^{1,2}_{\alpha',\beta'}(\mathbb{R}^N\setminus\overline{\mathbb{B}})$

Lemma

Let $N+\alpha>0$, $\beta>-1$ and $\alpha'=-2\alpha-\beta-4$. Then,

$$\mathcal{K}_1 : b^{1,2}_{\alpha,\beta}(\mathbb{B}\setminus\{0\}) \to b^{1,2}_{\alpha',\beta}(\mathbb{R}^N\setminus\overline{\mathbb{B}}) \text{ : isometry}$$

where $\mathcal{K}_1[u](x)=|x|^{2-N}u(x^*)$ and $x^*=\frac{x}{|x|^2}$. In addition, we have

$$R_{\mathbb{R}^N\setminus\overline{\mathbb{B}},1,\alpha',\beta}(x,y)=|x|^{2-N}|y|^{2-N}R_{\mathbb{B}\setminus\{0\},1,\alpha,\beta}(x^*,y^*)$$

for $x,y\in\mathbb{R}^N\setminus\overline{\mathbb{B}}$.

Lemma

If $\beta>-1$ and $-N<\alpha<N-4$, then

$$b^{1,2}_{\alpha,\beta}(\mathbb{B}\setminus\{0\})=\{f|_{\mathbb{B}\setminus\{0\}} : f\in b^{1,2}_{\alpha,\beta}(\mathbb{B})\}$$

Result for the weighted harmonic Bergman kernels

Theorem

Let $-N < \alpha < N - 2\beta - 4$ *and* $\beta \in \mathbb{N}_0$*. Then, we have*

$$\hat{R}_{\mathbb{B},1,\alpha,\beta}(x,y) = (-1)^{\beta} R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},1,\alpha,\beta}(x,y) \quad \text{for } x, y \in \mathbb{R}^N \setminus \overline{\mathbb{B}}.$$

A part of proof.

$$\begin{aligned} \hat{R}_{\mathbb{B},1,0,1}(x,y) &= \frac{d}{dt}\left(t^{\frac{N+\alpha}{2}+\beta+1} \hat{R}_{\mathbb{B},1,0,0}(tx,y) \right) \Big|_{t=1} \\ &= \frac{d}{dt}\left(t^{\frac{N+\alpha}{2}+\beta+1} R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},1,0,0}(tx,y) \right) \Big|_{t=1} \\ &= (\beta+1) R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},1,0,1}(x,y) \end{aligned}$$

□

Biharmonic Bergman kernel

$$b^{2,2}_{\alpha,\beta}(\mathbb{B}) \xrightarrow[\text{restriction}]{\text{certain condition}} b^{2,2}_{\alpha,\beta}(\mathbb{B}\setminus\{0\}) \xrightarrow[\mathcal{K}_2]{} b^{2,2}_{\alpha',\beta'}(\mathbb{R}^N\setminus\overline{\mathbb{B}})$$

where $\alpha' = N - 2\beta - 8$, $-N < \alpha < N - 2\beta - 8$ and $\mathcal{K}_2 f(x) := |x|^{4-N} f(x^*)$.

Lemma

If $\beta > -1$, $-N < \alpha < N - 2\beta - 8$ and $\alpha' = -8 - 2\beta - \alpha$, then

$$R_{\mathbb{R}^N\setminus\overline{\mathbb{B}},2,\alpha',\beta}(x,y) = |x|^{4-N}|y|^{4-N} R_{\mathbb{B},2,\alpha,\beta}(x^*,y^*)$$

for $x, y \in \mathbb{R}^N \setminus \overline{\mathbb{B}}$.

Main theorem

Theorem

Let $\beta \in \mathbb{N}$ *and* $-N < \alpha < N - 2\beta - 8$*. Then,*

$$\hat{R}_{\mathbb{B},2,\alpha,\beta}(x,y) = (-1)^{\beta} R_{\mathbb{R}^N \setminus \overline{\mathbb{B}},2,\alpha,\beta}(x,y) \quad \textit{for } x,y \in \mathbb{R}^N \setminus \overline{\mathbb{B}}.$$

Proof.

$$\overline{\mathrm{span}\{e_j^k(x), |x|^2 e_j^k(x)\}_{j,k}} = b^{2,2}_{\alpha,\beta}(\mathbb{B})$$

Hence, we can construct an orthonormal basis of $b^{2,2}_{\alpha,\beta}(\mathbb{B})$ and we have

$$\begin{aligned} R_{\mathbb{B},2,\alpha,\beta}(x,y) &= (\beta+2)R_{\mathbb{B},1,\alpha,\beta+2}(x,y) - (\beta+1)R_{\mathbb{B},1,\alpha,\beta+1}(x,y) \\ &\quad - (|x|^2+|y|^2)(\beta+2)R_{\mathbb{B},1,\alpha,\beta+2}(x,y) \\ &\quad + |x|^2|y|^2 \left((\beta+2)R_{\mathbb{B},1,\alpha,\beta+2} + (\beta+1)R_{\mathbb{B},1,\alpha+2,\beta+1}(x,y)\right) \end{aligned}$$

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Conclusion and Problems

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If $\beta \in \mathbb{N}_0$ and $-N < \alpha < N - 2\beta - 8$, then we have

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Problem

- For $m \geq 3$, the form of $R_{\mathbb{B},m,\alpha,\beta}(x,y)$.
- For $N + \alpha < 0$, $b^{1,2}_{\alpha,0}(\mathbb{B} \setminus \{0\})$.
- Harmonic Bergman kernel of an annular domain.

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